Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

CUBE COMPLEXES AND VIRTUAL FIBERING OF 3-MANIFOLDS

Tesi di Laurea Magistrale in Topologia

Relatore:
Chiar.mo Prof.
Stefano Francaviglia

Presentata da:
Lorenzo Ruffoni

I Sessione
Anno Accademico 2012/13

To Nicole

Preface

This thesis is about how geometry, group theory and topology interact on low-dimensional manifolds, with a special emphasis on the role of non-positive curvature. The situation is well-understood for surfaces; for instance, any closed orientable surface is homeomorphic to a sphere, a torus or a surface with more holes. Moreover the fundamental group is a complete invariant for surfaces, which means that two surfaces are homeomorphic if and only if they have isomorphic fundamental groups. As far as geometry is concerned, each of these surfaces carries a natural Riemannian structure with constant curvature k, where $k \in \{1, 0, -1\}$ and its precise value depends only on the topology of the surface (or, equivalently, on its fundamental group); the surface is called elliptic when $k = 1$, euclidean when $k = 0$, hyperbolic when $k = -1$, and this is the generic case.

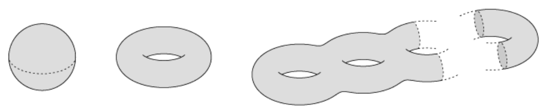

Figure 1: Elliptic, euclidean and hyperbolic surfaces

The 3-dimensional case is worse and better at the same time. It is worse since there are examples of non homeomorphic manifolds which have the same fundamental group (as some pairs of lens spaces), as well as examples of manifolds which do not admit Riemannian metrics of constant curvature (for example some surface bundles as $S^2 \times S^1$). In the early 1980's Thurston spotted a new way of thinking about geometric structures on manifolds in terms of group actions on some model spaces; this led him to define eight 3-dimensional model geometries and to conjecture that it is always possible to cut a 3-manifold in pieces which admit one of these geometries. This is roughly the statement of his celebrated Geometrization Conjecture. Notice that this is quite worse than the case of surfaces, first of all because we have to cut, and then because we have to tolerate five rather weird geometries.

One of the main achievements in this approach is due to Perelman, who in 2003 proved this conjecture. A consequence of his work is that the generic case is again that of hyperbolic geometry (the manifolds admitting the other seven geometries have been explicitly classified long before by Seifert). But this is even better than it looks, since in dimension 3 we have the deep Rigidity Theorem of Mostow, which roughly states that hyperbolic 3-manifolds are determined up to isometry by their fundamental group. This is quite remarkable because it means that geometric properties are actually topological invariants of the underlying manifold.

Hyperbolic 3-manifolds are among the most traditional spaces which display the typical features of what deserves the name of "non-positively" curved geometry and it is interesting that the algebraic and combinatorial objects one associates to them display such properties too, when these are interpreted in a suitable way. For instance, Gromov introduced the notion of hyperbolic group motivated by the example of the fundamental groups of hyperbolic manifolds. These are groups which come equipped with a representation as isometry groups of a certain associated non-positively curved space.

Many of the properties of these groups can be encoded in combinatorial structures constructed from some suitable collection of subgroups. These are the so called cube complexes, which are like simplicial complexes, but built from cubes instead of simplices, and are the main focus of this thesis, together with their application in 3-dimensional topology. As far as the geometry of codimension 1 subspaces (hyperplanes) is concerned, they have a much richer structure than simplicial complexes, essentially because you have two canonical ways of splitting a square into two equal parts, but no canonical way of splitting a triangle. This simple observation is the basis of the whole theory of cube complexes.

By a recent result of Kahn and Markovic on the existence of almost geodesic surfaces in a hyperbolic 3-manifold, the geometric properties of these manifolds allow us to cubulate them, i.e. to construct a cube complex with the same fundamental group; the manifold and the cube complex actually turn out to be homotopy equivalent, as an application of their being non-positively curved, but in general the dimension of the cube complex is much greater than 3. Remarkably, this is smarter than it sounds: Wise and Agol have shown in the last few years that it is indeed a great deal to pay some extra dimensions to get from a wild hyperbolic 3-manifold to a well organized cubical structure.

Their work has led to a proof of the following conjecture, which Thurston posed as question 16 in [Thu82]:

Theorem (Virtually Fibered Conjecture)**.** *A closed hyperbolic* 3*-manifold is virtually fibered, i.e. it has a finite index covering space which is a bundle over* S^1 *with fiber a surface.*

As hinted above, non hyperbolic geometric 3-manifolds (also known as Seifert manifolds) have been very well understood since a long time. To be more specific, they admit the structure of a fiber bundle with base a surface (actually a 2-dimensional orbifold) and fiber a circle, and can be classified according to this structure. Therefore a striking consequence of the proof of the above conjecture is that the world of 3-dimensional manifolds is "essentially" a fibered world, where "essentially" means up to a suitable decomposition in geometric pieces (following Thurston) and (possibly) up to finite covers.

Here is an outline of the material covered in the chapters in this thesis.

As described above, the hyperbolicity of manifolds induces non-positively curved phenomena in the algebraic and combinatorial objects we associate to them. To give a precise meaning to this sentence we need to introduce the necessary machinery to talk about curvature in abstract metric spaces, which we do in Chapter 1. We also introduce the right notion of equivalence under which these phenomena are preserved, i.e. quasi-isometry.

Groups arise naturally in geometric concrete examples as fundamental groups of topological spaces or isometry groups of metric spaces. In Chapter

2 we show that indeed any abstract (finitely presented) group can be realized as a fundamental group and as an isometry grou, of some suitably constructed spaces. We discuss applications of this point of view to the relationship between homotopy and cohomology of CW complexes and introduce the notion of hyperbolic group.

Chapter 3 is about cube complexes and their hyperplanes (= codimension 1 subspaces). We especially focus on some conditions on the way these hyperplanes sit and intersect inside the complex, which are known as speciality conditions. The interest in special cube complexes is due to their close relationship wiht right-angled Artin groups, which are a class of groups with a quite easy presentation and nice topological and geometric properties.

In order to understand how and why hyperbolic geometry turns out to be so fundamental in the study of 3-manifolds, in Chapter 4 we describe the classic decomposition techniques of 3-dimensional topology and the geometric program initiated by Thurston and closed by Perelman.

The final chapter shows how to apply the geometric (Chapter 1), algebraic (Chapter 2) and combinatorial (Chapter 3) techniques developed before to the study of virtual fibrations of closed hyperbolic 3-manifolds, which, by the reductions of Chapter 4, are the only case which is not fully understood. We present the work (especially by Wise and Agol) on how to go from a closed hyperbolic 3-manifold to a special cube complex and then to a virtual fibration of the manifold.

Introduzione

Questa tesi tratta delle interazioni tra geometria, teoria dei gruppi e topologia su varietà di dimensione bassa, con attenzione particolare al ruolo della curvatura non positiva. La situazione è ben nota nel caso delle superfici; ad esempio, ogni superficie chiusa orientabile è omeomorfa a una sfera, un toro o una superficie con più buchi. Inoltre il gruppo fondamentale è un invariante completo per le superfici, cioè due superfici sono omeomorfe se e solo se hanno lo stesso gruppo fondamentale. Per quanto riguarda la geometria, ciascuna di queste superfici ammette una naturale struttura Riemanniana a curvatura costante k, con $k \in \{1, 0, -1\}$, il cui preciso valore dipende solo dalla topologia della superficie (o, equivalentemente, dal suo gruppo fondamentale); la superficie si dice ellittica per $k = 1$, euclidea per $k = 0$ e iperbolica per $k = -1$, e questo è il caso generico.

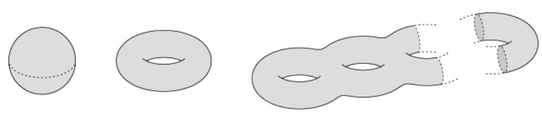

Figure 2: Superficie ellittica, euclidea e iperbolica

Il caso 3-dimensionale è allo stesso tempo migliore e peggiore. È peggiore in quanto ci sono esempi di varietà non omeomorfe che hanno lo stesso gruppo fondamentale (come certe coppie di spazi lenticolari), così come ci sono esempi di varietà che non ammettono metriche di Riemann a curvatura costante (ad esempio certi fibrati con fibra una superficie, come $S^2 \times S^1$). All'inizio degli anni 1980 Thurston ha individuato un nuovo modo di pensare le strutture geometriche su varietà in termini di azioni di gruppi su certi spazi modello; questa impostazione lo ha condotto a definire otto geometrie modello 3-dimensionali e a congetturare che sia sempre possibile tagliare una 3-varietà in pezzi che ammettano una di queste geometrie. Questo è all'incirca l'enunciato della sua celebre Congettura di Geometrizzazione. Si noti che questo caso è alquanto peggiore di quello delle superfici, prima di tutto perché dobbiamo eseguire una decomposizione, e poi perché dobbiamo tollerare cinque geometrie piuttosto strane.

Uno dei successi principali in questo approccio è dovuto a Perelman, che nel 2003 ha provato tale congettura. Una conseguenza del suo lavoro è che il caso generico è ancora quello della geometria iperbolica (le varietà che ammettono le altre sette geometrie sono state classificate tempo fa da Seifert). Ciò è anche migliore di quel che sembra, in quanto in dimensione 3 abbiamo il profondo Teorema di Rigidità di Mostow, che essenzialmente afferma che le 3-varietà iperboliche sono determinate a meno di isometria dal loro gruppo fondamentale. Questo è davvero notevole, in quanto implica che le proprietà geometriche sono in effetti invarianti topologici della varietà soggiacente.

Le 3-varietà iperboliche sono tra gli spazi più tradizionali che manifestano le caratteristiche tipiche di ciò che merita il nome di geometria "non positivamente curvata" ed è interessante che gli oggetti algebrici e combinatori che associamo loro manifestano anch'essi tali proprietà. Ad esempio Gromov ha introdotto la nozione di gruppo iperbolico, motivata dall'esempio del gruppo fondamentale di una varietà iperbolica. Si tratta di gruppi che ammettono una rappresentazione come gruppi di isometrie di opportuni spazi non positivamente curvati.

Molte delle proprietà di questi gruppi possono essere codificate con strutture combinatorie costruite a partire da opportune collezioni di sottogruppi. Questi sono i cosiddetti complessi cubici, che sono simili ai complessi simpliciali, ma costruiti a partire da cubi anziché da simplessi, e sono il fulcro principale di questa tesi, assieme alla loro applicazione in topologia 3-dimensionale. Per quanto riguarda la geometria dei sottospazi di codimensione 1 (iperpiani), essi godono di una struttura molto più ricca dei complessi simpliciali, essenzialmente perché ci sono due modi canonici di dividere a metà un quadrato, ma non ci sono modi canonici di dividere a metà un triangolo. Questa semplice osservazione è alla base dell'intera teoria dei complessi cubici.

Grazie ad un recente risultato di Kahn e Markovic sull'esistenza di superfici quasi geodetiche in una 3-varietà iperbolica, le proprietà geometriche di queste varietà ci permettono di cubularle, cioè di costruire un complesso cubico con lo stesso gruppo fondamentale; la varietà e il complesso cubico risultano proprio essere omotopicamente equivalenti, il che segue dal fatto che sono entrambi spazi non positivamente curvati, ma in generale la dimensione del complesso cubico è di molto superiore a 3. Sorprendentemente ciò non è così male come sembra: Wise e Agol hanno mostrato negli ultimi anni che è in effetti un buon affare pagare qualche dimensione in più per poter passare da una complicata 3-varietà iperbolica ad una struttura cubica ben organizzata.

Il loro lavoro ha portato ad una prova della seguente congettura, che Thurston ha posto come domanda 16 in [Thu82]:

Teorema (Congettura di Fibrazione Virtuale)**.** *Una 3-varietà iperbolica chiusa è virtualmente fibrata, cioè ha un rivestimento di indice finito che è un fibrato su S^1 con fibra una superficie.*

Come accennato sopra, le 3-varietà con geometrie non iperboliche (note anche come varietà di Seifert) sono ben comprese da lungo tempo. Più esplicitamente, esse ammettono una struttura di fibrato con base una superficie (in realtà un orbifold 2-dimensionale) e fibra un cerchio, e possono essere classificate secondo tale struttura. Una notevole conseguenza della dimostrazione della suddetta congettura è dunque che il mondo delle 3-varietà è un mondo "essenzialmente" fibrato, ove "essenzialmente" significa a meno di opportune decomposizioni in pezzi geometrici (seguendo Thurston) ed (eventualmente) a meno di rivestimenti finiti.

Segue una breve descrizione del materiale trattato nei vari capitoli di questa tesi.

Come descritto sopra, l'iperbolicità delle varietà induce geometrie non positivamente curvate sugli oggetti algebrici e combinatori che ad esse associamo. Per dare un significato preciso a questa affermazione occorre introdurre la tecnologia necessaria per parlare di curvatura in spazi metrici astratti, cosa che facciamo nel Capitolo 1. Inoltre introduciamo l'appropriata nozione di equivalenza sotto cui tali fenomeni sono preservati, cioè la quasi-isometria.

I gruppi si presentano naturalmente in esempi geometrici concreti come gruppi fondamentali di spazi topologici o come gruppi di isometrie di spazi metrici. Nel Capitolo 2 mostriamo che in effetti ogni gruppo astratto (finitamente presentato) può essere realizzato come gruppo fondamentale e come gruppo di isometrie di opportuni spazi. Presentiamo anche alcune applicazioni di questo punto di vista alla relazione tra omotopia e coomologia di CW complessi e introduciamo la nozione di gruppo iperbolico.

Il Capitolo 3 è dedicato ai complessi cubici e ai loro iperpiani (=sottospazi di codimensione 1). Ci concentriamo specialmente su alcune condizioni sul modo in cui questi iperpiani giacciono e si intersecano dentro il complesso, che sono note come condizioni di specialità. L'interesse nei complessi cubici speciali risiede nella loro stretta relazione coi gruppi di Artin ad angolo retto, che sono una classe di gruppi con una presentazione piuttosto semplice e interessanti proprietà topologiche e geometriche.

Per capire in che modo e perché la geometria iperbolica si riveli così fondamentale nello studio delle 3-varietà, nel Capitolo 4 descriviamo le classiche tecniche di decomposizione della topologia 3-dimensionale e il programma geometrico iniziato da Thurston e concluso da Perelman.

L'ultimo capitolo mostra come applicare le tecniche geometriche (Capitolo 1), algebriche (Capitolo 2) e combinatorie (Capitolo 3) sviluppate precedentemente allo studio delle fibrazioni virtuali di 3-varietà iperboliche chiuse, le quali, grazie alle riduzioni del Capitolo 4, costituiscono l'unico caso ancora non pienamente compreso. Presentiamo il lavoro (specialmente dovuto a Wise e Agol) su come passare da una 3-varietà iperbolica chiusa a un complesso cubico speciale e quindi a una fibrazione virtuale della varietà.

Contents

List of Figures

Chapter 1

Non-Positively Curved Geometries

The Uniformization Theorem for surfaces shows that hyperbolic geometry is the most common in dimension 2. In Chapter 4 we will describe the work of Thurston and Perelman on the geometrization of 3-manifolds, which reveals that the same is true in dimension 3. This is a nice fact, since non-positively curved shapes exhibit a lot of interesting features from a geometric and homotopical point of view.

Some of these properties do not actually rely to the locally euclidean structure of hyperbolic manifolds, and in this chapter we will describe a generalization of hyperbolic geometry to abstract metric spaces. This will be applied to groups and cube complexes in the following chapters.

1.1 Preliminaries about Metric Spaces

In this section we collect the basics definition about metric spaces, stressing the analogy with the case of Riemannian manifolds.

Definition 1.1.1. Let (X, d) be a metric space. A geodesic from $x \in X$ to $y \in X$ is a continuous path $\gamma : [0, L] \to X$ from x to y such that for all $s, t \in [0, L]$ we have $d(\gamma(s), \gamma(t)) = |s - t|$.

Example 1.1.2. If (M, g) is a Riemannian manifold, then length minimizing Riemannian geodesics are geodesics in the above sense, by definition of the

Riemannian distance d_g associated to the Riemannian metric g. Notice that not all geodesics on a sphere are geodesics in the above sense (since some are not length minimizing).

Definition 1.1.3. A metric space (X, d) is said to be

- a geodesic space if every couple of points is joined by a geodesic;
- a uniquely geodesic space if every couple of points is joined by a unique geodesic; we also say X is R-uniquely geodesic if its balls of radius at most R are uniquely geodesic;
- a length space if $\forall\ x, y \in X$ we have that $d(x, y) = \inf\{L(\gamma) | \gamma$ is a geodesic joining x and $y\}$; if there are no geodesics joining x and y we agree to say that $d(x, y) = \infty$;
- a proper space if closed balls are compact;
- a cocompact space if it admits a compact subset K whose translates under the action of $\mathrm{Isom}(X)$ cover the whole space.

A geodesic space is a length space, right away from the above definitions. To see how in general things can go wrong we consider a few examples.

Example 1.1.4. S^1 with the distance induced by the euclidean distance of $\mathbb{R}^2$ (i.e. the chord distance) is not a length space (and so it is neither geodesic). But if we equip it with the Riemannian metric induced by $\mathbb{R}^2$ (i.e. the arc length) and consider the associated path metric then it becomes a geodesic space.

Example 1.1.5. The punctured plane $\mathbb{R}^2 \setminus \{0\}$ is a length space which is not geodesic.

The point in the last example is the failure of completeness. We have the following classical result.

Theorem 1.1.6 (Hopf, Rinow)**.** *A connected, complete and locally compact length space is a geodesic space.*

For a detailed proof see [BrH99], Proposition 3.7. This result of course applies to complete connected Riemannian manifolds when considered as metric spaces with respect to the path metric (see [BrH99], Corollary I.3.20). Notice anyway that in general these spaces can fail to be uniquely geodesic, as the example of spheres shows.

We now introduce a family of metric spaces which come from classic Riemannian manifolds of constant curvature and will be used as model spaces to induce a notion of curvature on more general metric spaces.

Definition 1.1.7. For each real number k let M_k^n denote the following metric space:

- if $k = 0$ then $M_k^n = \mathbb{E}^n$ with the euclidean metric;
- if $k > 0$ then M_k^n is obtained from $\mathbb{S}^n$ by multiplying the distance function by $k^{-1/2}$;
- if $k < 0$ then M_k^n is obtained from $\mathbb{H}^n$ by multiplying the distance function by $(-k)^{-1/2}$.

We also denote by D_k the diameter of M_k^n, i.e. $D_k = \frac{\pi}{k}$ for $k > 0$ and $D_k = \infty$ otherwise.

In the following sections we will introduce the machinery needed to speak of curvature in an abstract metric space. Two different approaches (and the relationship between them) are discussed.

Both approaches are based on some kind of consideration about triangles, so we advance this definition.

Definition 1.1.8. Let (X, d) be a metric space. A geodesic triangle Δ in X consists of three vertices $p, q, r \in X$ and a choice of three geodesics which will be denoted $[p, q]$, $[q, r]$ and $[r, p]$ and called the edges of the triangle. The triangle will also be denoted by $\Delta([p, q], [q, r], [r, p])$ or just $\Delta(p, q, r)$ (even if this may cause some confusion when the space is not uniquely geodesic).

1.2 CAT(k) Condition

The approach we discuss in this section stems from the work of Cartan, Alexandrov and Toponogov and is based on the comparison with the model spaces defined in 1.1.7. In the sequel of this section (X, d) will denote a geodesic metric space.

Definition 1.2.1. Let $\Delta = \Delta(x_1, x_2, x_3) \subset X$ be a geodesic triangle. A k-comparison triangle is a geodesic triangle $\overline{\Delta} = \overline{\Delta}(\overline{x}_1, \overline{x}_2, \overline{x}_3) \subset M_k^2$ such that $\overline{d}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$, where $\overline{d}$ denotes the distance in M_k^2. A point $\overline{x} \in [\overline{x}_i, \overline{x}_j]$ is called a comparison point for a point $x \in [x_i, x_j]$ if $d(x_i, x) = \overline{d}(\overline{x}_i, \overline{x})$.

We of course need the following result, which is Lemma I.2.14 in [BrH99].

Lemma 1.2.2. *Let $\Delta \subset X$ be a geodesic triangle and $k \in \mathbb{R}$. If its perimeter is less than $2D_k$ then there exists a k-comparison triangle and it is unique up to isometry of M_k^2.*

We are now ready to give the main definition of this section:

Definition 1.2.3. Let $k \in \mathbb{R}$. A geodesic triangle $\Delta \subset X$ is said to satisfy the CAT(k) condition if its perimeter is less than $2D_k$ and if for all $x, y \in \Delta$ and for all comparison points $\overline{x}, \overline{y}$ in a k-comparison triangle $\overline{\Delta}$ we have that $d(x, y) \leq d(\overline{x}, \overline{y})$.

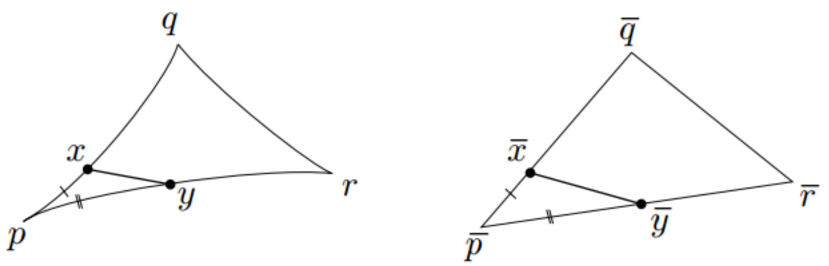

Figure 1.1: The CAT(k) condition

Definition 1.2.4. A space X is called a CAT(k) space if it is geodesic and its geodesic triangles satisfy the CAT(k) condition. It is said to have curvature $\leq k$ if it is locally CAT(k).

Example 1.2.5. Riemannian manifolds are CAT(k) in this sense if and only if all of their sectional curvatures are bounded by k. See Theorem II.1.1A.6 in [BrH99] for details.

Having a bound on the (curvature of the) geometry of a metric space turns out to give some unexpected properties, both from a geometric and homotopy-theoretic point of view. The following result is Proposition II.1.4 in [BrH99].

Lemma 1.2.6. *Let X be a CAT(k) space. Then*

1. *X is D_k-uniquely geodesic,*
2. *balls of radius at most D_k are contractible.*

Proof. Let $x, y \in X$ such that $d(x, y) \leq D_k$; since X is geodesic by definition, we can find at least one geodesic γ joining x to y. Let γ' be another such geodesic and let z and z' be points respectively on γ an γ' such that $d(x, z) = d(x, z')$. We denote by δ and η the two geodesics in which γ is divided by z. Then we consider the geodesic triangle Δ with vertices x, z, y and edges δ, η and γ'. But now we observe that any k-comparison triangle degenerates in a segments, thus the comparison points for z and z' coincide. From the CAT(k) inequality we obtain that $z = z'$, therefore γ' coincides with γ and 1 is proved.

Let B a ball of radius $R < D_k$ centered at x; for any $y \in B$ by 1 we have a unique geodesic γ_y joining x and y. Then we may define a map $B \times [0, 1] \to B$ which sends the point (y, t) to the unique point on γ_y at distance $td(x, y)$ from y. To check that this defines a continuous retraction of the ball onto its center x one has to verify that γ_y depends continuously on y; this is again an application of the CAT(k) inequality, see [BrH99] for details. $\square$

We will be mostly interested in non-positively curved spaces, i.e. (locally) CAT(0) spaces, since many of the algebraic and combinatorial objects that we will associate to a hyperbolic 3-manifold will admit this kind of geometry. One of the nice things about non-positive curvature is the following fact, which readily follows from the above lemma and the fact that $D_k = \infty$ when $k \leq 0$.

Corollary 1.2.7. *For any $k \leq 0$, a CAT(k) space is contractible.*

This allows to extend many classic results about non-positively curved Riemannian manifolds to this more general context. For example, from a generalization of the Cartan-Hadamard Theorem one can prove the following useful result, which is Proposition II.4.14 in [BrH99].

Theorem 1.2.8. *Let X and Y be complete connected metric spaces; suppose that X is locally a length space and that Y is non-positively curved. Let $f : X \to Y$ be a locally isometric embedding. Then*

- *X is non-positively curved,*
- *the induced map $f_* : \pi_1(X) \to \pi_1(Y)$ is injective.*

1.3 Quasi-isometries

In this section we introduce a notion of metric equivalence which is strictly weaker than isometry, but is the right thing to consider for the application to group theory we will develop in the next chapter.

Definition 1.3.1. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a (not necessarily continuous) function . Then we give the following definitions:

- if $\exists\ \lambda \geq 1, \varepsilon \geq 0$ such that $\forall\ p, q \in X$ we have
$$\frac{1}{\lambda} d_X(p, q) - \varepsilon \leq d_Y(f(p), f(q)) \leq \lambda d_X(p, q) + \varepsilon$$
then we say that f is a (λ, ε)-quasi-isometric embedding ;
- if $\exists\ \lambda \geq 1, \varepsilon \geq 0$ such that f is a (λ, ε)-quasi-isometric embedding and $\exists\ C \geq 0$ such that $\forall\ y \in Y$ we have that y belongs to the C-neighbourhood of the image of f, then we say that f is a (λ, ε)-quasi-isometry.

When we do not care about the constants involved, we just say that f is a quasi-isometric embedding (respectively, a quasi-isometry) if it is a (λ, ε)-quasi-isometric embedding (respectively, a (λ, ε)-quasi-isometry) for some $\lambda \geq 1, \varepsilon \geq 0$.

Here are some (trivial) examples.

Example 1.3.2. A metric space is quasi-isometric to a point if and only if it is bounded.

Example 1.3.3. The inclusion of a subspace $Y \hookrightarrow X$ is a quasi-isometric embedding. If $\exists\ C \geq 0$ such that $\forall\ x \in X\ \exists\ y \in Y$ such that $d(x, y) \leq C$, then Y is said to be a quasi-dense subspace. This is equivalent to saying that the inclusion is a quasi-isometry. As an interesting example we have that $\mathbb{Z}$ and $\mathbb{R}$ are quasi-isometric (when equipped with the euclidean distance).

This definition is rich enough to prove some basic results.

Lemma 1.3.4. *If $f : X \to Y$ is a quasi-isometry, then we can find a quasi-isometry $g : Y \to X$ and a $k \geq 0$ such that $\forall\ x \in X\ d_X(gf(x), x) \leq k$ and $\forall\ y \in Y\ d_Y(fg(y), y) \leq k$. Such a map is called a quasi-inverse of f.*

Proof. By definition, we can choose a pair of constants λ and ε for which f is a (λ, ε)-quasi-isometry and a constant $C \geq 0$ such that $\forall\ y \in Y$ we have $d_Y(y, f(X)) \leq C$. For each $y \in Y$ we can find some $x_y \in X$ such that $d_Y(y, f(x_y)) \leq C$ and we define $g(y) := x_y$ and then we check the desired properties.

Let $y_1, y_2 \in Y$ and let $x_i := g(y_i), i = 1, 2$. By the triangle inequality we have

$$d_Y(y_1, y_2) \leq d_X(y_1, f(x_1)) + d_X(f(x_1), f(x_2)) + d_X(f(x_2), y_2) \leq \dots$$

and then from the choice of x_i and the fact that f is a (λ, ε)-quasi-isometry we obtain

$$\dots \leq \lambda d_X(x_1, x_2) + \varepsilon + 2C = \lambda d_X(g(y_1), g(y_2)) + \varepsilon + 2C$$

which, since $\lambda \geq 1$, is equivalent to

$$\frac{1}{\lambda} d_Y(y_1, y_2) - \frac{2C + \varepsilon}{\lambda} \leq d_X(g(y_1), g(y_2))$$

Reasoning in the same way but in the other direction we can prove that

$$d_X(g(y_1), g(y_2)) \leq \lambda d_Y(y_1, y_2) + \lambda(2C + \varepsilon)$$

Since $\lambda \geq 1$ we also have that $\dfrac{2C+\varepsilon}{\lambda} \leq \lambda(2C+\varepsilon)$ and so g is a $(\lambda, \lambda(2C+\varepsilon))$-quasi-isometric embedding.

Next we observe that by construction we have that $d_Y(y, fg(y)) \leq C$. On the other hand if $x \in X$ then let $x' := gf(x)$; we have

$$\frac{1}{\lambda} d_X(x, x') - \varepsilon \leq d_Y(f(x), f(x')) \leq C$$

therefore we have $d_X(x, gf(x)) \leq \lambda(C+\varepsilon)$. These are the desired inequalities; this also implies that $\forall\ x \in X\ d_X(x, g(Y)) \leq \lambda(C+\varepsilon)$, so g is a quasi-isometry. □

Lemma 1.3.5. *The composition of quasi-isometries is a quasi-isometry.*

Proof. Let $f : X \to Y$ be a (λ, ε)-quasi-isometry with $C \leq 0$ such that $\forall\ y \in Y\ d_Y(y, f(X)) \leq C$ and let $g : X' \to Y'$ be a (λ', ε')-quasi-isometry with $C' \leq 0$ such that $\forall\ y \in Y\ d_Y(y, f(X)) \leq C'$. First of all we see that if $z \in Z$ then $\exists\ y \in Y$ such that $d_Z(z, g(y)) \leq C'$ and $\exists\ x \in X$ such that $d_Y(y, f(x)) \leq C$; but then $d_Z(z, gf(x)) \leq d_Z(z, g(y)) + d_Z(g(y), gf(x)) \leq C' + \lambda' d_Y(y, f(x)) + \varepsilon' \leq C' + \lambda' C + \varepsilon'$. Moreover we check that $\forall\ x, x' \in X$

$$d_Z(gf(x), gf(x')) \leq \lambda' d_Y(f(x), f(x')) + \varepsilon' \leq$$

$$\leq \lambda'(\lambda d_X(x, y) + \varepsilon) + \varepsilon' = \lambda\lambda' d_X(x, y) + \lambda'\varepsilon + \varepsilon'$$

and on the other hand

$$d_Z(gf(x), gf(x')) \geq \frac{1}{\lambda'} d_Y(f(x), f(x')) - \varepsilon' \geq$$

$$\geq \frac{1}{\lambda'}\left(\frac{1}{\lambda} d_X(x, x') - \varepsilon\right) - \varepsilon' \geq \frac{1}{\lambda\lambda'} d_X(x, x') - (\lambda'\varepsilon + \varepsilon')$$

where the last inequality follows because $\lambda' \geq 1$. This proves that $fg : X \to Y$ is a $(\lambda\lambda', \lambda'\varepsilon + \varepsilon')$-quasi-isometry. □

Since the identity map is obviously a quasi-isometry (indeed it is an isometry), we get the following result:

Corollary 1.3.6. *Being quasi-isometric is an equivalence relation among metric spaces.*

1.4 δ-Hyperbolicity

We now present an approach to curvature in abstract metric spaces which is due to Gromov (see [Gro87], but also [BrH99]). This is slightly different from the CAT(k) approach of section 1.2, since no comparison with model geometries is involved here: the idea of curvature is given in intrinsic terms.

Definition 1.4.1. Let (X, d) be a metric space and $\delta \geq 0$. We say that

- a geodesic triangle Δ is δ-slim if each of its edges is contained in the δ-neighbourhood of the union of the other two edges;

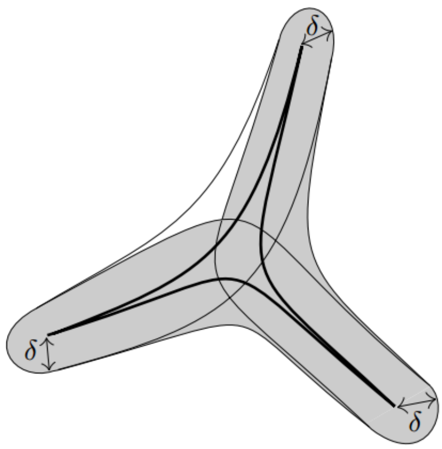

Figure 1.2: A δ-slim triangle

- X is δ-hyperbolic if it is geodesic and every geodesic triangle is δ-slim;
- X is hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

Example 1.4.2. Trees are 0-hyperbolic.

Example 1.4.3. As one may expect, the hyperbolic plane $\mathbb{H}^2$ is of course δ-hyperbolic. This follows from the fact that the area of a hyperbolic triangle is bounded above by π. This of course generalizes to $\mathbb{H}^n$.

Example 1.4.4. The euclidean plane $\mathbb{E}^2$ is not hyperbolic. To see this just take the triangle with one vertex at the origin O and the other two with coordinates $(0, t)$ and $(t, 0)$. Then let P be the midpoint of the hypotenuse;

we have $d(P,O) = \dfrac{t}{\sqrt{2}} \xrightarrow[t\to\infty]{} \infty$, so no δ can verify the condition in the definition. The same applies to $\mathbb{E}^n$ for $n \geq 3$.

Example 1.4.5. Each bounded space X is hyperbolic: just take $\delta = \mathrm{diam} X$. A surprising consequence is that $\mathbb{S}^2$ (and any other sphere) is hyperbolic. This theory is not discriminating in the realm of bounded spaces.

Remark 1.4.6. As the above examples show, even if the notion of hyperbolicity encodes some form of non-positive curvature, however it is not quite the same thing as being CAT(0). For instance $\mathbb{E}^2$ is CAT(0) but not hyperbolic. Interestingly enough, this is the only essential difference between the two notions, as the following theorem (Theorem III.H.1.5 in [BrH99]) says.

Theorem 1.4.7 (Flat Plane Theorem)**.** *A proper cocompact CAT(0) space is hyperbolic if and only if it does not contain an isometrically embedded copy of* $\mathbb{E}^2$*.*

Being hyperbolic is of course an isometric invariant, but for our applications to group theory we will need an invariance with regards to the weaker notion of quasi-isometry introduced in the previous section. We have the following result (Theorem III.H.1.9 in [BrH99]).

Theorem 1.4.8. *Let X and Y be geodesic metric spaces and $f : X \to Y$ a (λ, ε)-quasi-isometric embedding. If Y is δ-hyperbolic, then $\exists\ \widehat{\delta} = \widehat{\delta}(\delta, \lambda, \varepsilon)$ such that X is $\widehat{\delta}$-hyperbolic.*

In the next two paragraphs we introduce two fundamental objects in the study of metric spaces, which encode the behaviour of the space at infinity and turn out to be quasi-isometric invariants in a lot of interesting cases. This invariance is a way to make more precise the slogan according to which "quasi-isometries preserve the large-scale geometry of the spaces". The construction could be given for quite general metric spaces; however we will be interested in these objects only for δ-hyperbolic spaces.

1.4.1 The Boundary at ∞

The Poincaré disk model for $\mathbb{H}^n$ is homeomorphic to the open unit ball in $\mathbb{R}^n$. A natural compactification of this space is obtained by adding the boundary sphere $\mathbb{S}^{n-1}$. This is a fundamental property of classical hyperbolic geometry: for example it provides ideal triangles and a smart way to classify hyperbolic isometries.

It turns out that this feature does not rely on the particular homeorphism with euclidean ball, but is an intrinsic property of the non-positive curved geometry of $\mathbb{H}^n$. Here we describe a generalization of this phenomenon to general δ-hyperbolic spaces.

Definition 1.4.9. A geodesic ray in a metric space (X, d) is a continuous path $\gamma : [0, +\infty[$ such that $s, t \in [0, +\infty[$ we have $d(\gamma(s), \gamma(t)) = |s - t|$.

Definition 1.4.10. Two geodesic rays γ and γ' are said to be asymptotic if $\sup_t d(\gamma(t), \gamma'(t))$ is finite. This is an equivalence relation between geodesic rays which we denote by $\sim_\infty$; we also denote by $\gamma(\infty)$ the equivalence class of a geodesic ray. We then define

$$\partial X := {}^{\{\text{geodesic rays in } X\}}\!\big/_{\sim_\infty}$$

and we call it the boundary at infinity of X. $\overline{X} := X \cup \partial X$ is called the bordification of X.

Remark 1.4.11. This construction can be carried out for any metric space, but is best suited for proper geodesic hyperbolic spaces. For this class of spaces ∂X can be topologized so that quasi-isometric spaces have homeomorphic boundaries (for details see [BrH99], chapter III.H.3).

1.4.2 The Ends

The notion we are going to introduce should encode the behaviour of a space "outside an arbitrarily large compact subset".

Definition 1.4.12. A proper ray in a topological space X is a continuous path $r : [0, +\infty[\to X$ such that the preimage of a compact set is compact.

Definition 1.4.13. Two proper rays γ and γ' are said to converge to the same end if $\forall\, K \subset X$ compact $\exists\, N \geq 0$ such that $\gamma([N, +\infty[)$ and $\gamma'([N, +\infty[)$ lie in the same path component of $X \setminus K$. This is an equivalence relation between proper rays which we denote by $\sim_{end}$; we also denote by $end(\gamma)$ the equivalence class of a proper ray. We then define

$$Ends(X) := {}^{\{\text{proper rays in } X\}}\!\big/_{\sim_{end}}$$

and we call it the space of ends of X. We denote by $e(X)$ its cardinality.

This set is usually topologized by defining a notion of convergence for ends.

Definition 1.4.14. We say that $end(\gamma_n)$ converges to $end(\gamma)$ for $n \to \infty$ if for every compact $K \subset X$ we can find integers $N_n \geq 0$ such that $\gamma_n([N_n, +\infty[)$ and $\gamma([N_n, +\infty[)$ lie in the same path component of $X \setminus K$, at least for n large enough.

We then have the following result (see Proposition I.8.29 in [BrH99]).

Theorem 1.4.15. *Any quasi-isometry $f : X \to Y$ between proper geodesic spaces induces an homeomorphism $f_{end} : Ends(X) \to Ends(Y)$.*

The manifest analogy between the construction of the boundary at infinity and the space of ends is not accidental. Of course asymptotic rays converge to the same end. Indeed one has the following result.

Proposition 1.4.16. *Let X be a geodesic proper hyperbolic space. The map $\partial X \to Ends(X)$ is continuous and the fibers are the connected components of ∂X.*

Chapter 2

The Geometric and Topological Approach to Group Theory

From an historical point of view, group theory has found in the many branches of geometry one of the main sources of examples and ideas. But groups are traditionally studied as abstract structures and groups arising in geometry and topology are usually considered "just" particular examples. On the contrary, it can be shown that every group has geometric essence, even if it is given in purely abstract terms.

The aim of this chapter is to give the previous sentence some precise meaning, for example showing that every group can be realized as a group of isometries of a suitable metric space, or as the fundamental group of a suitable topological space. The study of algebraic properties of groups through the geometric properties of these spaces is known as Geometric Group Theory.

We present the main tools and results of this theory. In the end we discuss a special class of groups introduced by Gromov in the 1980's which have given a huge boost to the theory and are also of great interest in the study of low-dimensional manifolds.

2.1 Groups as Fundamental Groups

Throughout this chapter let G be a group and fix some finite presentation $G = \langle g_\alpha | r_\beta \rangle$. It is well known that every group has a presentation (e.g. it is isomorphic to a quotient of the free group over itself), but this is quite trivial and useless. Even if some aspects of the theory apply to the general setting, in the following we consider only finitely presented groups, since this is the class of groups that arise in the study of low-dimensional manifolds and which we will be concerned with.

In this section we construct some topological spaces with fundamental group isomorphic to G and derive some results that will be useful in the following chapters.

2.1.1 Presentation Complex

Let X be a connected 1-dimensional CW complex. Then $\pi_1(X)$ is a free group. Suppose we attach a 2-cell via some attaching map $\varphi : S^1 \to X$ and we call the resulting space Y; we have a natural inclusion $X \hookrightarrow Y$. Of course $\varphi(S^1)$ gives a loop in Y and a standard application[1] of Seifert-van Kampen theorem shows that $\pi_1(Y) \cong {}^{\pi_1(X)}\!/\!_{N(\varphi(S^1))}$, where $N(\varphi(S^1))$ is the normal closure of $\varphi(S^1)$ in $\pi_1(X)$.

In other words, attaching 2-cells to a graph introduces relations in its fundamental group. This allows us to prove the following theorem.

Theorem 2.1.1. *Every group G is the fundamental group of a finite 2-dimensional CW complex.*

Proof. Let $G = \langle g_\alpha | r_\beta \rangle$ be a presentation of G. Let X be a bouquet of oriented circles, one for each g_α. Each relation r_β is a finite word in these generators (and their inverses) as $r_\beta = g_{\alpha_1}^{\pm 1} \dots g_{\alpha_p}^{\pm 1}$. Then we glue a 2-cell e^2_β via an attaching map defined in this way: subdivide the boundary of e^2_β in p edges $a_1 \dots a_p$ and send a_i to the loop labelled $g_{\alpha_i}^{\pm 1}$ (where $g_{\alpha_i}^{-1}$ is just g_{α_i} with the opposite orientation). Do this for each relation and call X_G the resulting space. From the previous discussion it follows that $\pi_1(X_G) \cong G$. $\square$

[1] For a detailed proof see Proposition 1.26 in [Hat02].

Definition 2.1.2. The complex X_G constructed in the previous proof is called the Presentation Complex associated to the presentation $G = \langle g_\alpha | r_\beta \rangle$ of G.

Example 2.1.3. Let G be a cyclic group of order n; a presentation is given by $G = \langle g | g^n \rangle$, so X_G is obtained from a circle by glueing a disk via a map whose restriction to the boundary is a map of degree n. For example for $n = 2$ we get $X_G \cong \mathbb{RP}^2$. But for $n \geq 3$ we never get a smooth surface.

Remark 2.1.4. The previous example shows that in general we cannot expect X_G to be locally euclidean, i.e. to be a topological manifold. For the sake of completeness, we report the following result, which says that if you accept to pay some extra dimension then you can always realize G in a nicer way.

Theorem 2.1.5. *Every finitely presented group G is the fundamental group of a 4-dimensional compact connected smooth manifold.*

This theorem, combined with the undecidability of the Isomorphism Problem for groups, is the reason why manifolds of dimension $n \geq 4$ cannot be effectively classified. The same theorem is false in dimension 3.

The following example shows a key feature of this theory: groups that look strange from a purely algebraic point of view, may have a quite simple presentation complex which encodes a lot of extra structure, which may be at first not visible in the combinatorial data of a presentation.

Example 2.1.6. Let $G = \langle a_1, b_1 \ldots, a_{2n}, b_{2n} | \prod_{i=1}^{n} [a_i, b_i] \rangle$. Then X_G is the closed orientable surface of genus n. This follows from the fact that such a surface can be obtained by glueing the edges a regular polygon with $4n$ edges (in $\mathbb{R}^2$ if $n = 2$ or $\mathbb{H}^2$ if $n \geq 3$).

Remark 2.1.7. One can ask whether the complex X_G, that a priori depends on the chosen presentation, is actually an invariant of G. The answer turns out to be no in a rather strong sense: presentation complexes associated to different presentations of G can even be not homotopically equivalent, as the following example shows.

Example 2.1.8. Let G be the trivial group; a presentation is given by $G = \langle \varnothing | \varnothing \rangle$ and the associated complex is just a point. But G can also be presented by $G = \langle a | a, a^{-1} \rangle$, and the complex associated to this presentation is S^2.

For now, we content ourselves with the following result which gives a partial answer to the question of how much a group G determines the complexes X_G associated to different presentations. See Proposition 1B.9 in [Hat02] for a proof.

Theorem 2.1.9. *Let $f : G \to H$ be a homomorphism of groups. Then we can find a presentation complex associated to some presentation of each group and a map $\varphi : X_G \to X_H$ such that $f = \pi_1(\varphi)$.*

The point is that the map φ will not in general be unique, and this let different presentations give non homotopically equivalent complexes. The next section deals with this problem.

2.1.2 Eilenberg-MacLane Spaces

In this section we refine the construction of the presentation complex to achieve better results.

Definition 2.1.10. Let G be a group. An Eilenberg-MacLane space for G is a path connected topological space X with contractible universal cover and such that $\pi_1(X) = G$. We also say that X is a $K(G, 1)$.

Remark 2.1.11. From the long exact sequence associated to the universal covering map, we see that in particular all higher homotopy groups are trivial. So a $K(G, 1)$ is nicer than a presentation complex from the point of view of homotopy theory, since all the higher dimensional homotopy has been killed; on the other hand we pay our debt with the fact that typically a $K(G, 1)$ has a cell structure with cell in high dimensions (even in infinitely many dimensions sometimes).

Example 2.1.12. S^1 is a $K(\mathbb{Z}, 1)$ and more generally a graph is a $K(G, 1)$ for G a free group, since the universal cover is a tree.

Example 2.1.13. A closed hyperbolic surface S is a $K(\pi_1(S), 1)$, because its universal cover is $\mathbb{H}^2$. The same is true for higher dimensional closed hyperbolic manifolds.

Example 2.1.14. In the previous section we have seen S^2 and $\mathbb{RP}^2$ arising as presentation complexes. They are not $K(G, 1)$ for their fundamental groups.

Remark 2.1.15. It can be proved that every group admits a $K(G, 1)$, for example by attaching higher dimensional cells to any presentation complex for G; however the general construction produces infinite dimensional complexes even if G admits some easier $K(G, 1)$, as in the case of $\mathbb{Z}$. In the following we will not need this full generality, since we will be able to find $K(G, 1)$ "by hands".

The main reason to switch from presentation complexes to Eilenberg-MacLane spaces is the fact that they are homotopically rigid, i.e. do not show the ambiguity of example 2.1.8. We have a result analogous to Theorem 2.1.9 but enriched by a universal property which guarantees uniqueness.

Theorem 2.1.16. *Let X be a connected CW complex, Y a $K(G, 1)$ and $f : \pi_1(X, x_0) \to \pi_1(Y, y_0) = G$ a homomorphism. Then $\exists\ \varphi : (X, x_0) \to (Y, y_0)$, which is unique up to homotopy relative to the basepoint.*

A proof of this result is available in [Hat02]. It is then easy to prove that:

Corollary 2.1.17. *The homotopy type of a CW complex $K(G, 1)$ is determined by G.*

Proof. Let X, Y be two $K(G, 1)$ with a CW complex structure; their fundamental groups are thus isomorphic (and both isomorphic to G). Let $f : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ a couple of inverse homomorphisms (i.e. $f = g^{-1}$ and vice versa). The previous theorem gives a pair of maps $\varphi : (X, x_0) \to (Y, y_0)$ and $\psi : (X, x_0) \to (Y, y_0)$ inducing f and g respectively. But then $\varphi\psi$ and $\psi\varphi$ induce the identity on the fundamental groups and so (again by the theorem) are homotopic (relatively to basepoints) to the identities on each space. Thus φ and ψ give a homotopy equivalence between X and Y. $\square$

We now want to discuss two applications of the previous constructions: the first is mainly algebraic, whereas the second is of geometric interest.

Remark 2.1.18. The previous result gives a smart way to carry the whole machinery of homological algebra from the category of topological spaces to that of groups: just define the (co)homology groups of a group G to be the (co)homology groups of any $K(G, 1)$:

$$H_i(G) := H_i(K(G, 1))$$

$$H^i(G) := H^i(K(G, 1))$$

this is well-posed since all $K(G, 1)$ are homotopy equivalent. This turns out to be equivalent to the algebraic approach to group (co)homology via Tor and Ext functors.

Remark 2.1.19. The other application is about the connection between homotopy and cohomology of CW complexes. Let X, Y be CW complexes and $\langle X, Y\rangle$ denote the set (no additional structure is involved here) of basepoint-preserving homotopy classes of maps $X \to Y$. A classical result in algebraic topology (see e.g. Theorem 4.57 in [Hat02]) states that for any abelian group G there is a canonical bijection (of sets)

$$\langle X, K(G, 1)\rangle \longleftrightarrow H^1(X, G)$$

where $H^1(X, G)$ denotes singular cohomology with coefficients in G. The case $G = \mathbb{Z}$ is really nice: here we can take S^1 as $K(G, 1)$ and obtain the bijection

$$\langle X, S^1\rangle \longleftrightarrow H^1(X)$$

Now clearly every map $X \to S^1$ induces a morphism $\pi_1(X) \to \mathbb{Z}$. But since S^1 is a $K(\mathbb{Z}, 1)$ the theorem says that the converse is also true, that is every morphism $\pi_1(X) \to \mathbb{Z}$ is induced by a map $X \to S^1$ unique up to basepoint-preserving homotopy. This gives a bijection

$$H^1(X) \longleftrightarrow \langle X, S^1\rangle \longleftrightarrow \mathrm{Hom}(\pi_1(X), \mathbb{Z})$$

that we will exploit in the following chapters.

2.2 Groups as Isometry Groups

In this section we introduce spaces which have a more geometric flavour: they come equipped with natural metrics and isometric G-actions.

2.2.1 Cayley Graph

Let S be a fixed set of generators for G.

Definition 2.2.1. The Cayley Graph associated to G and to S is the graph $\mathrm{Cay}(G, S)$ defined in this way:

- the vertices of $\mathrm{Cay}(G, S)$ are the elements of G;
- two vertices $x, y \in G$ span an edge if and only if $\exists\ s \in S$ such that $y = xs$; in this case we label this edge by s and orient it from x to y.

Remark 2.2.2. We usually want S to be closed under inversion, so that we can just speak of "words in the generators" instead of having to consider "words in the generators and their inverses". Anyway this can produce some redundancy in the construction of the Cayley graph; we adopt the following conventions:

- If $e \in S$ (where e denotes the identity of G), then we get a trivial loop at each point, which we agree to erase.
- When $s, s^{-1} \in S$, for each $g \in G$ we get an edge labelled s from g to gs and an edge labelled s^{-1} from gs to g; they of course carry the same information, so we draw just one of them. This amounts to make a choice of one generator for each couple $s, s^{-1} \in S$ (notice that this works only if $s \neq s^{-1}$); any such choice gives the same (undirected) graph.
- If some generator has order 2 (i.e. $s = s^{-1}$), then the previous choice cannot be done: in this case we agree to represent it as an undirected edge .

With these conventions, adding to a set of generators the inverses of its elements (as well as the identity), does not change the resulting Cayley graph (as long as we do not care about orientations of the edges).

Remark 2.2.3. $\mathrm{Cay}(G,S)$ is a connected graph with 0-skeleton identified with G itself and there is a natural action of G on $\mathrm{Cay}(G,S)^0$ given by left multiplication

$$G \times \mathrm{Cay}(G)^0 \to \mathrm{Cay}(G)^0, g, x \mapsto gx$$

This action is free and transitive on $\mathrm{Cay}(G,S)^0$ and can be extended to a simplicial action to the 1-cells by setting

$$g, [x, xs] \mapsto [gx, gxs]$$

If the generating set is finite, the action is also cocompact, i.e. $\exists\ K \subset \mathrm{Cay}(G,S)$ compact such that $G.K = \mathrm{Cay}(G,S)$; namely K is given by vertices corresponding to the identity of G and the generators in S together with all edges between them; equivalently, the quotient ${\mathrm{Cay}(G,S)}\big/_{G}$ is a compact complex.

Example 2.2.4. When the standard generating sets are understood, $\mathrm{Cay}(\mathbb{Z}_n)$ is just a loop subdivided in n arcs, $\mathrm{Cay}(\mathbb{Z}^n)$ is the integer lattice in $\mathbb{R}^n$ and, if $G = F_n$ is the free group on n generators, then $\mathrm{Cay}(G)$ is the tree in which each vertex has $2n$ outgoing edges.

There is a standard way to turn a graph into a metric space, just setting the length of each edge.

Definition 2.2.5. The Cayley metric d_S on $\mathrm{Cay}(G,S)$ is obtained declaring each edge isometric to the unit interval $[0,1] \subset \mathbb{R}$ and then considering the path-metric.

Remark 2.2.6. There is a correspondence between edge-paths in $\mathrm{Cay}(G,S)$ and words in G (with respect to the generating set S), since edges of $\mathrm{Cay}(G,S)$ are labelled by generators of G. The nice thing about the Cayley metric is that for each $g \in G$ the length of the shortest word representing g in the generators in S equals the distance $d_S(g,e)$ of g from the identity element in the Cayley graph associated to the generating set S.

Definition 2.2.7. The word metric on G (with respect to the generating set S) is defined to be the restriction of the Cayley metric to the vertex set, which is identified with G by construction. In this way G itself is turned into a metric space.

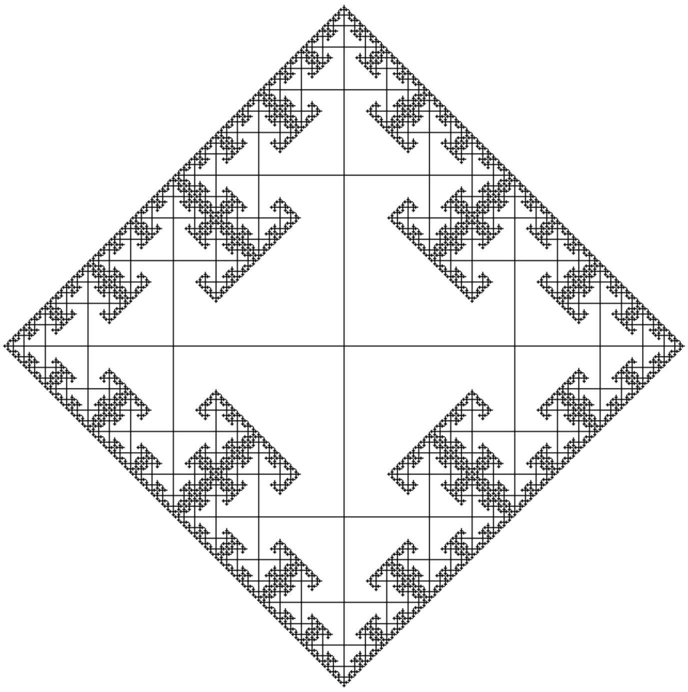

Figure 2.1: The Cayley Graph of F_2, the free group on two generators

As one may expect we have the following result

Proposition 2.2.8. *G acts on Cay(G, S) by isometries of the Cayley metric.*

Proof. This actions sends adjacent vertices to adjacent vertices □

Notice that G also acts by right multiplication, but this is not in general an isometric action, since adjacent vertices may be sent to very distant ones. Of course left multiplication gives an isometric action of G on itself with respect to the word metric.

Before leaving this section, it is natural to ask how much Cay(G, S) depends on the chosen generating set. As for Eilenberg-MacLane spaces, different generating set may yield non homeomorphic objects, but we can find a suitable kind of equivalence.

Proposition 2.2.9. *If G is a finitely generated group and S, S' are different generating sets then Cay(G, S) and Cay(G, S') are quasi-isometric.*

Proof. Here we consider four metric spaces: (Cay(G, S),d), (Cay(G, S'),d'), (G, d) and (G, d'), that is the two Cayley graphs with their word metrics and the two metric spaces obtained from G itself equipped with the respective word metrics. The inclusion of (G, d) into (Cay(G, S),d) is a quasi-isometric embedding (it is an isometric embedding indeed); moreover by the definition of the Cayley metric, each point in (Cay(G, S),d) lies in the $\frac{1}{2}$-neighbourhood of the 0-skeleton (which is identified with G itself), therefore the inclusion is indeed a quasi-isometry. From the discussion in 1.3, it is then enough to show that (G, d) and (G, d') are quasi-isometric.

Each generator $s_i \in S$ can be written as a word in S'. Let l_i denote the minimal length of such a word and let $K_1 := \max l_i$, which is well posed since S is finite. It is then clear from the previous discussion that $d_{S'}(g, e) \leq K_1 d_S(g, e)$. Exchanging the roles of S and S' define K_2 and obtain $d_S(g, e) \leq K_2 d_{S'}(g, e)$. Finally set $K := \max\{K_1, K_2\}$. This proves that the identity map on G induces a $(K, 0)$-quasi-isometry between (G, d) and (G, d') (which is actually a bilipschitz equivalence, i.e. we do not need any additive constant). □

Remark 2.2.10. It follows from the previous proof that each group has an associated metric space well-defined only up to quasi-isometry, i.e. a quasi-isometry type. This is the right thing to consider, since we cannot expect to obtain an isometry type in general, as the following example shows in a quite dramatic way.

Example 2.2.11. Consider the group of integers $\mathbb{Z}$; this is naturally generated by 1. As discussed in the example 2.2.4, the Cayley graph associated to this generating set is an infinite chain, which we can identify with the real line (compare also example 1.3.3).

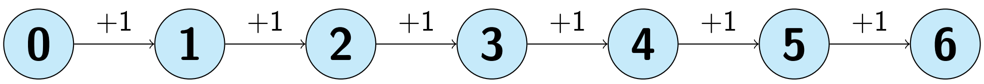

Figure 2.2: (A portion of) the Cayley Graph of $\mathbb{Z}$ with respect to $\{1\}$

But we can as well generate $\mathbb{Z}$ with $\{2, 3\}$ (or with any other couple of coprime numbers by Bézout's Identity), and this would give a strongly different Cayley graph; the next picture shows an example of a portion of what we would get.

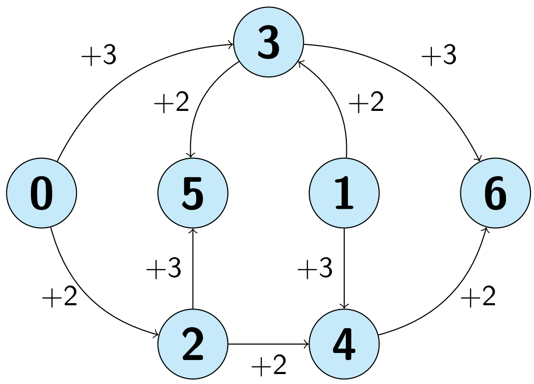

Figure 2.3: (A portion of) the Cayley Graph of $\mathbb{Z}$ with respect to $\{2,3\}$

As you can see we get closed loops, which exemplifies the fact that graphs arising from different presentation can not only be non isometric, but even not homotopy equivalent. One of the main features of the notion of quasi-isometry is that it does not require continuity.

2.2.2 Cayley Complex

In the constructions of the previous section we have focused on the generating part of a presentation of G, neglecting relations; now we want to incorporate them into this construction. This will lead to a connection to presentation complexes.

Let $G = \langle S|R\rangle = \langle s_1,\ldots,s_n|r_1,\ldots,r_p\rangle$ be a presentation for G and $\mathrm{Cay}(G,S)$ the associated Cayley graph. The relations in R give rise to loops in $\mathrm{Cay}(G,S)$ which give generators for π_1 $(\mathrm{Cay}(G,S))$. We want to make these loops nullhomotopic to obtain a simply connected complex with a nice G-action and which still contains $\mathrm{Cay}(G,S)$ as a subcomplex. To do this we can glue a 2-cell along each of these loops. This certainly yields a simply connected complex with a natural G-action obtained extending the action on $\mathrm{Cay}(G,S)$ in a cellular way, but the action is not nice enough, as the following example shows.

Example 2.2.12. Take $\mathbb{Z}_n = \langle g | g^n \rangle$. Then the above construction gives a complex which has one 2-cell with the boundary divided into n arcs (it is just $\mathrm{Cay}(\mathbb{Z}_n, \{[1]\})$). The action of $\mathbb{Z}_n$ on the boundary is by rotations, therefore the extension to the complex is not free because it fixes some point in the interior of the cell. The general situation of a non-free action is not so different: if we have a non trivial stabilizer of some inner point of a 2-cell, then its action always induces a permutation of the boundary vertices and so the action on the whole cell is that of a dihedral group.

One way to avoid this problem is to consider that each relation gives a loop based at *any* vertex of $\mathrm{Cay}(G, S)$ and to glue a 2-cell to each of these loops. In this way, whenever we see a loop of length k we actually consider k different loops based at the k different vertices on the loops and glue k different 2-cells identifying their boundaries in $\mathrm{Cay}(G, S)$. When we act with an element that stabilizes the loop, the action on the boundary is the one described above, but in the interior we go from one cell to another. In this way we remove fixed points.

Definition 2.2.13. The 2-complex obtained by the above construction is called the Cayley complex associated to the presentation $G = \langle S | R \rangle$.

Example 2.2.14. Take $\mathbb{Z}_n = \langle g | g^n \rangle$. The Cayley complex is obtained by n copies of the unit disk glued along their boundaries, which come equipped with a subdivision in n arcs which descend to an analogous subdivision of the unique boundary in the quotient. The action of g sends a cell to another cell after a rotation of $\frac{2\pi}{n}$. For example for $n = 2$ this is just the covering map $S^2 \to \mathbb{RP}^2 = X_{\mathbb{Z}_2}$. Notice that if we glue just one cell to $\mathrm{Cay}(\mathbb{Z}_2, \{[1]\})$ then we get a disk (not a sphere) and the quotient map is not a covering map since it has a cone singularity of order 2 at the origin.

The situation described in the previous example is quite general.

Proposition 2.2.15. *The Cayley complex is the universal cover of the presentation complex X_G associated to the same presentation of G.*

Proof. The above discussion proves that the action of G on this complex is free and properly discontinuous, and the orbit space is just X_G; thus it is a covering space of X_G. Moreover it is simply connected by construction, so it is the universal cover. $\square$

We now quote the following theorem by Nielsen and Schreier, which shows an example of the way in which algebraic properties can be deduced from geometric and topological considerations about the spaces we have constructed in this chapter.

Theorem 2.2.16. *A subgroup H of a free group G is free.*

Proof. Take X_G as a bouquet of circles. Since no relations are involved the universal cover is just the tree $\mathrm{Cay}(G)$. A subgroup H of G is associated to an intermediate covering space $\mathrm{Cay}(G) \to Y_H \to X_G$. But then Y_H is necessarily a graph, so $H = \pi_1(Y_H)$ must be free. $\square$

2.3 Hyperbolic Groups

As an application to the material presented so far, we present a class of groups introduced by Gromov in [Gro87] which happens to be of great interest in low-dimensional topology as we will see in the last chapter.

Definition 2.3.1. A finitely generated group G is hyperbolic (also Gromov-hyperbolic or word-hyperbolic) if its Cayley graph with respect to some generating set is δ-hyperbolic (fore some δ) when equipped with the Cayley metric.

Remark 2.3.2. Notice that this definition does not depend on the choice of the generating set, but the precise value of δ does. Indeed from 2.2.9 we know that Cayley graphs with respect to different generating set are quasi-isometric and from 1.4.8 we know that being hyperbolic is a quasi-isometric invariant.

Example 2.3.3. From the examples in 1.4.2 and 2.2.4 we readily have that

- finite groups and free groups are hyperbolic;
- free abelian groups are not hyperbolic;
- any group with a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ cannot be hyperbolic.

The last example is quite significant and motivates the following criterion, which will be useful in later chapters.

Theorem 2.3.4. *Let G be a group acting properly and cocompactly by isometries on a CAT(0) space X. Then G is hyperbolic if and only if X does not contain an isometrically embedded copy of $\mathbb{E}^2$.*

This is Theorem III.Γ.3.1 in [BrH99], which we refer to for details. Notice the analogy with the Flat Plane Theorem (see 1.4.7).

Remark 2.3.5. From what we have said so far, it follows that every group *is* a quasi-isometric type of geodesic metric spaces. G is hyperbolic if and only if this quasi-isometric type is hyperbolic. In 1.4.1 we have constructed a boundary at infinity for hyperbolic spaces and said that quasi-isometric proper geodesic spaces have homeomorphic boundary (with respect to some suitable topology). As a result we can unambiguously talk about *the* boundary at infinity ∂G of a hyperbolic group G; notice that the Cayley graphs associated to a finitely generated group are proper metric spaces, since the unit ball centered at the origin has finitely many vertices. The same reasoning of course applies to the space of ends introduced in 1.4.2, therefore the following definition is well posed.

Definition 2.3.6. Let G a hyperbolic group. We define the boundary at infinity of G and the space of ends of G to be

$$\partial G := \partial \mathrm{Cay}(G,S) \quad \text{and} \quad Ends(G) := Ends(\mathrm{Cay}(G,S))$$

for some (finite) generating set S. We also define $e(G)$ as the cardinality of $Ends(G)$.

Chapter 3

Special Cube Complexes

The aim of this chapter is to introduce a class of CW complexes which have proved really useful in the study of 3-manifolds, since they provide a smart way of embedding fundamental groups into well-organized groups. The standard reference for the whole material is taken from [HaW08], with the exception of the proof of Theorem 3.3.8 which has been made self-contained, avoiding any appeal to the metric properties of these complexes. A discussion of the original approach is given in the last section.

3.1 Hyperplanes and Walls in Cube Complexes

Let I denote the interval $[-1, 1] \subset \mathbb{R}$ and I^n its n-fold cartesian power, which we call n-cube.

Definition 3.1.1. A cube complex is a CW complex which is obtained by glueing cubes via isometries of their faces. In other words, the attaching map of each n-cell (i.e. n-cube) is defined on ∂I^n and its restriction to an $(n-1)$-face of ∂I^n is given by an isometry of that face with I^{n-1} composed with an $(n-1)$-cell of X.

Remark 3.1.2. Notice that this definition allows two n-cells to be glued along two arbitrary faces (which are not necessarily 1-codimensional), as long as they have the same dimension; for example we can glue two squares at one point or along an edge. We can also assemble a square and a segment by

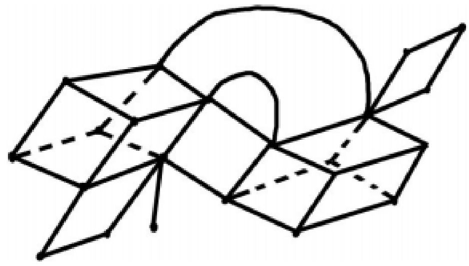

Figure 3.1: A cube complex

identifying two points in their 0-skeleton; this allows for complexes which may have different local maximal dimension in different points, i.e. we can have one point which is contained in a n-dimensional cell and another point such that each cell which contains it has dimension $k < n$. Another thing to observe is that we do not require the attaching maps to be injective on the boundary of the cubes; for example an n-torus is a cube complex: this can be seen considering it as a quotient of I^n in the usual way.

The main feature of cube complexes is the availability of canonical subcomplexes of (local) codimension 1, which we now define.

Definition 3.1.3. A midcube of I^n is a subspace of the form $M = \{x \in I^n \mid x_i = 0\}$ for some i and the edges of I^n which are orthogonal to it are called its dual edges.

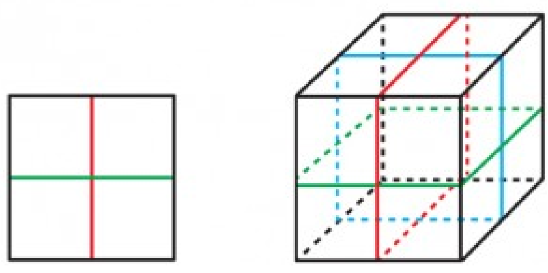

Figure 3.2: Midcubes in I^2 and I^3

Remark 3.1.4. The crucial properties of midcubes are the following

- if a midcube $M \subset I^n$ intersects a $(n-1)$-face $F \subset I^n$, then the intersection $M \cap F$ is a midcube N of F;
- vice versa, for each midcube N in a $(n-1)$-face $F \subset I^n$ there is a unique midcube $M \subset I^n$ such that $M \cap F = N$; this means there is a unique way to extend a midcube of a face to a midcube of the whole cube.

By induction on the dimension of the cells, the previous properties hold for any face of the cube, not necessarily 1-codimensional.

This allows us to propagate in a canonical way a midcube of a cell to adjacent cells; notice that this property does not hold in simplicial complexes. What is obtained after a "maximal propagation" is a locally 1-codimensional subspace, which can be thought of as a "hypersurface" inside X or, as we will say, a hyperplane. But we want an abstract way to define these objects.

Definition 3.1.5. Given a cube complex X we construct a new cube complex $\mathcal{H}(X)$, called hyperplane complex of X, as follows: the n-cubes of $\mathcal{H}(X)$ are midcubes of the $(n+1)$-cubes of X and the attaching maps in $\mathcal{H}(X)$ are obtained in the obvious way by restriction of those of X to midcubes. A hyperplane of X is by definition a connected component of $\mathcal{H}(X)$.

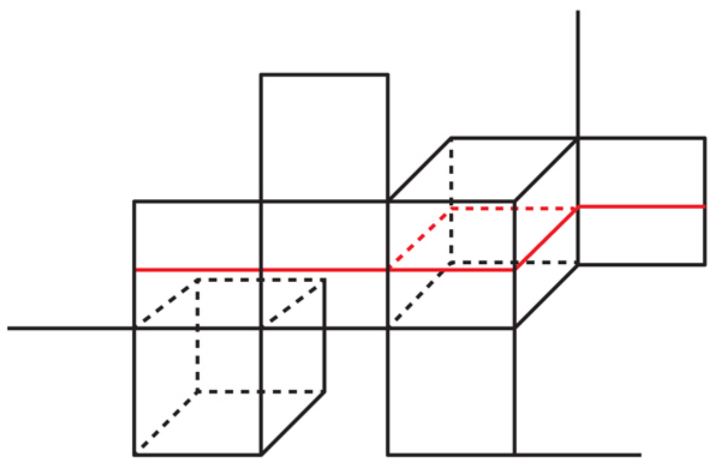

Figure 3.3: A hyperplane in a cube complex

Being locally 1-codimensional allows hyperplanes to be identified with their 1-dimensional complement; this gives rise to a fundamental duality in a cube complex, which we know describe.

Definition 3.1.6. Let X be a cube complex.

- An edge $a \in X^1$ is said to be dual to an hyperplane H of X if its midpoint is a vertex of H (in $\mathcal{H}(X)$); this is equivalent to saying that H intersects a (in X).
- Two edges of X are said to be elementary parallel if they appear as opposite edges of a square. We call parallelism the equivalence relation among edges of X generated by elementary parallelisms. If $a \in X^1$, its parallelism class is denoted $W(a)$ and called the wall through a.

The following proposition establishes the desired duality.

Proposition 3.1.7. *There is a bijective correspondence between hyperplanes and walls of X, obtained associating at each hyperplane the set of dual edges and to each wall the unique hyperplane to wich it is dual.*

The uniqueness in the previous proposition is guaranteed by the fact (stressed above) that it is possible to extend midcubes in a unique way from a cube to the adjacent ones. This result is very useful, since it allows to translate conditions about the hyperplanes about conditions about the walls, which live only in the 2-skeleton of X and are thus easier to deal with in proofs.

We need just one more technical definition.

Definition 3.1.8. The link of a point v in a cube complex X is the complex $lk(v, X)$ obtained by taking a point for each edge that contains the point and in which $k + 1$ points span a k-simplex if and only if there is a cube of dimension $k + 1$ which contains the corresponding edges in its boundary. A geometric realization of this complex is given by intersecting the cube complex with a small sphere centered at the point v.

In general the link of a vertex carries only the structure of a CW-complex.

Definition 3.1.9. A cube complex is simple if the link of each vertex is a simplicial complex.

This in particular means that two squares cannot meet in a pair of consecutive edges, because otherwise we would see a bigon in the link of the vertex in the middle, which is forbidden in the definition of simplicial complex. We will consider only simple cube complexes in the following.

3.2 Speciality Conditions

We now turn our interest to a class of cube complexes in which a few kinds of pathologies are forbidden. These pathologies are about the way hyperplanes are immersed in the complex; it will be useful to have a definition in term of hyperplanes themselves and a dual one about walls.

Definition 3.2.1. Let X be a cube complex, H, K two hyperplanes in X and W, Z their dual walls. We say that:

1. H selfintersects if it contains at least two midcubes from the same cube of X. Equivalently there are two edges of W which appear as adjacent edges of some square of X. H is said to be embedded if it does not selfintersect. We also say that a wall is selfintersecting or embedded if its dual hyperplane is selfintersecting or embedded.

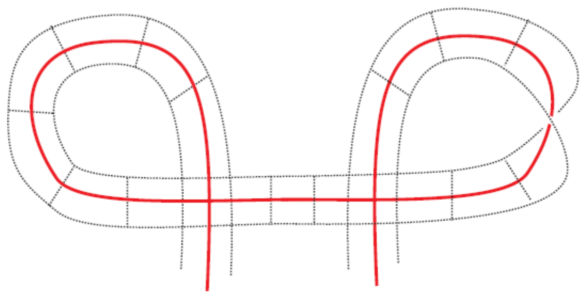

Figure 3.4: A selfintersecting hyperplane

2. An embedded hyperplane H is 2-sided if its normal bundle (i.e. the union of open cubes which contain the midcubes of H) is trivial, that is isomorphic to $H \times I$. Equivalently it is possibble to orient coherently all the edges of W.

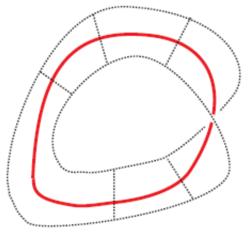

Figure 3.5: A 1-sided hyperplane

3. H and K osculate if "their normal bundle are tangent"; more precisely, we say H and K osculate at (v, a, b) if $a \in W$, $b \in Z$ and they intersect at the vertex $v \in X$, but there is no cube of X that contains both a and b (otherwise this would give a configuration of intersection). If H and K are 2-sided, the osculation is said to be direct or indirect if the orientations induced on v respectively agree or disagree. We also say that H selfosculates at (v, a, b) if $a, b \in W$ and they intersect at the vertex $v \in X$, but there is no cube of X that contains both a and b (otherwise this would give a configuration of selfintersection); as before we distinguish between direct and indirect selfosculation according to how orientations are induced on the common point.

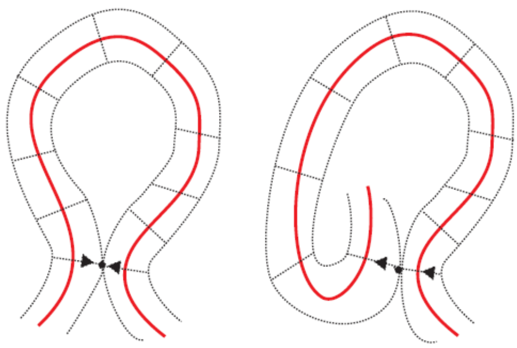

Figure 3.6: A directly selfosculating hyperplane (left) and an indirectly selfosculating hyperplane (right)

4. H and K inter-osculate if there is a cube in which they intersect and a

point (not in that cube) in which they osculate (without intersecting).

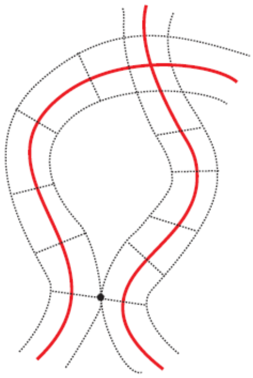

Figure 3.7: A pair of interosculating hyperplanes

We now are ready to give the main definition.

Definition 3.2.2. Let X be a cube complex. Then

- X is special if it is simple, each hyperplane is embedded and non directly selfosculating and there are no interosculating hyperplanes;
- X is A-special it is special and if each hyperplane is 2-sided.

Remark 3.2.3. Notice that since all the speciality conditions can be expressed in terms of edges and squares, everything here depends only on the 2-skeleton of X. Therefore X is (A-)special if and only if X^2 is (A-)special.

Remark 3.2.4. We give these two different definitions since we will associate to a closed hyperbolic 3-manifold a cube complex which in general is just special, but the theory of cube complexes needs A-speciality to express all of its power. We will see that, in a suitable sense, A-speciality can always be recovered from speciality alone.

3.2.1 Virtual Equivalence

In this section we prove the equivalence of the speciality conditions up to finite covers.

Definition 3.2.5. Let X, Y be simple cube complexes.

- A map $f : X \to Y$ is a combinatorial map if for each n-cell $\varphi_X : I^n \to X$ there exist a n-cell $\varphi_Y : I^n \to Y$ and an isometry $j : I^n \to I^n$ such that $f \circ \varphi_X = \varphi_Y \circ j$.

- A combinatorial map $f : X \to Y$ is an immersion if the maps induced on links $f_v : lk(v, X) \to lk(f(v), Y)$ are injective.

- An immersion $f : X \to Y$ is a local isometry if $\forall\ v \in X^0$ we have that $f_v(lk(v, X))$ is a full subcomplex of $lk(f(v), Y)$, i.e. if each simplex of $lk(f(v), Y)$ whose vertices are in $f_v(lk(v, X))$ is itself in $f_v(lk(v, X))$.

Remark 3.2.6. This terminology is reminiscent of that of Riemannian geometry: a combinatorial immersion is something like a smooth map with injective differential, and having full subcomplexes is something like having surjective differential. This analogy is not accidental: it is actually possible to turn cube complexes into nice metric spaces; our treatment could be carried out without appealing to this fact. However the original discussion in [HaW08] relies on this property, so we will spend a few words on it at the end of the chapter (see 3.4).

The next lemma is about stability of the speciality conditions under combinatorial maps.

Lemma 3.2.7. *Let $f : X \to Y$ be a combinatorial map between cube complexes. Then:*

1. *Y has embedded (resp. 2-sided) hyperplanes $\Rightarrow$ X has embedded (resp. 2-sided) hyperplanes;*

2. *suppose that either Y has embedded hyperplanes and f is an immersion, or that $f|_{X^2} : X^2 \to Y^2$ is a local isometry; then if Y has no direct selfosculations, then the same holds for X;*

3. *Y has embedded hyperplanes and has no interosculations and $f|_{X^2} : X^2 \to Y^2$ is a local isometry $\Rightarrow$ X has no interosculations.*

Proof. Since f is combinatorial, it sends edges and squares in X to edges and squares in Y, so it preserves elementary parallelism and sends walls in walls.

1. Suppose X has a wall W which selfintersects or is 2-sided; then $f(W)$ would be (contained in) a wall of Y which would selfintersect or be 2-sided, and this is absurd.

2. Suppose an oriented wall W of X directly selfosculates in $(v, \overrightarrow{a}, \overrightarrow{b})$; then in Y we have a wall V which passes through $f(\overrightarrow{a})$ and $f(\overrightarrow{b})$. Since f is a local immersion on X^2, we have $f(\overrightarrow{a}) \neq f(\overrightarrow{b})$; this means that V either selfintersects or directly selfosculates in $(f(v), f(\overrightarrow{a}), f(\overrightarrow{b}))$, but both options are absurd under the hypothesis.

3. Let V and W be wall of X which interosculate and let $(v, \overrightarrow{a}, \overrightarrow{b})$ be the configuration of osculation. The wall through $f(\overrightarrow{a})$ intersects the one through $f(\overrightarrow{b})$, so they must be distinct; if $f(\overrightarrow{a})$ and $f(\overrightarrow{b})$ were adjacent in $lk(f(v), Y)$ then $\overrightarrow{a}$ and $\overrightarrow{b}$ should be adjacent in $lk(v, X)$ because f is a local isometry; but this does not happen, by definition of interosculation. Therefore the two walls of Y osculate in $(f(v), f(\overrightarrow{a}), f(\overrightarrow{b}))$, which is absurd.

□

Since a covering map is of course a local isometry, we have the following result.

Corollary 3.2.8. *Each covering complex of a special (resp. A-special) complex is still special (risp. A-special).*

In the proof of the next lemma we will exploit a combinatorial approach to the homotopy properties of our cube complex.

Remark 3.2.9. We observe that, by cellular approximation, every path in a cube complex is homotopic to an edge-path in the 1-skeleton and each homotopy between edge-paths can be realized by a sequence of elementary homotopies taking place in some square. Elementary homotopies (fixing extremities) of an edge-path $x_1 \dots x_n$ in a square can only take one of these two forms:

1. $x_1 \dots x_{i-1} x_i x_i^{-1} x_{i+1} \dots x_n \simeq x_1 \dots x_{i-1} x_{i+1} \dots x_n$
2. $x_1 \dots x_i x_j \dots x_n \simeq x_1 \dots x_h x_k \dots x_n$ whenever $x_i x_j x_k^{-1} x_h^{-1}$ is the (oriented) perimeter of some square of X

The following result establishes the virtual equivalence (i.e. up to a finite cover) of the two definitions of speciality.

Lemma 3.2.10. *Let X be a cube complex with a finite number of walls and with no selfintersecting walls. Then $\exists\, X'$ finite cover of X whose hyperplanes are 2-sided.*

Proof. Let $W_1, \dots, W_n$ be the walls of X e let $\gamma : [0,1] \to X^1$ an edge-path. By the previous description of elementary homotopies (fixing extremities), the parity of intersection of γ with each W_j is preserved by these homotopies; therefore we have a morphism $\lambda_j : \pi_1(X) \to \mathbb{Z}_2$ whose kernel has finite index. Since the number of walls is finite,

$$H := \bigcap_{j=1}^{n} \ker(\lambda_j)$$

is a finite index subgroup, so the associated cover $p : X' \to X$ has finite degree.

Now suppose there is an edge $\overrightarrow{a}$ of X' which is parallel to its opposite $\overleftarrow{a}$ and let $\{\overrightarrow{a} = \overrightarrow{a}_1, \overrightarrow{a}_2, \dots, \overrightarrow{a}_m = \overleftarrow{a}\}$ be a sequence of elementary parallel edges which realizes the parallelism between $\overrightarrow{a}$ and $\overleftarrow{a}$. Let Q_k be the square in which the elementary parallelism between $\overrightarrow{a}_k$ and $\overrightarrow{a}_{k+1}$ takes place and let $\overrightarrow{b}_k$ the edge of Q_k such that $i(\overrightarrow{b}_k) = t(\overrightarrow{a}_k)$ and $t(\overrightarrow{b}_k) = t(\overrightarrow{a}_{k+1})$, where i and t denote the initial and terminal point of an edge with respect to the chosen orientation. In particular we have $t(\overrightarrow{b}_m) = t(\overrightarrow{a}_m) = t(\overleftarrow{a}) = i(\overrightarrow{a})$ which means that $\gamma := (\overrightarrow{a}, \overrightarrow{b}_1, \dots, \overrightarrow{b}_m)$ is a closed edge-path.

Then $p \circ \gamma \in H$ and by construction of H we have $\lambda_j(p \circ \gamma) = 0$ for each $j = 1, \dots, n$, in particular for that j such that $W_j = W(p(a))$; as a consequence $\#(\{p(a), p(b_1), \dots, p(b_m)\} \cap W_j)$ is even. On the other hand by construction $W_j \cap W(p(b_k)) \neq \varnothing$ and so $p(b_k) \notin W_j$ since X has embedded hyperplanes and this implies that $(\{p(a), p(b_1), \dots, p(b_m)\} \cap W_j) = \{p(a)\}$. This is absurd because it should have an even number of elements. $\square$

Theorem 3.2.11. *Let X be a special cube complex with a finite number of walls (e.g. let X be compact and special). Then $\exists\ X'$ finite cover of X that is A-special.*

Proof. From Lemma 3.2.10 we know there is a finite cover X' of X with 2-sided walls. Moreover from Corollary 3.2.8 we can conclude that X' is also special, so it is A-special indeed. □

3.3 Virtual Embedding in RAAGs

In the introduction to this chapter we have suggested that cube complexes are useful to embed fundamental groups in well-organized groups. In this section we describe the groups of this class and the details of the construction.

The main ideas are due to Haglund and Wise (see [HaW08]); their proof exploits a natural metric available on cube complex, but then rests on some deep results in CAT(k) geoemtry by Gromov, Cartan and Hadamard. Here we propose an elementary and self contained proof.

3.3.1 Right-Angled Artin Groups

We begin introducing the class of groups we are interested in.

Definition 3.3.1. Let Γ be a simplicial graph. The right-angled Artin group (RAAG in the following) associated to Γ is the group with the presentation:

$$A(\Gamma) = \langle x_i \in \Gamma^0 | [x_i, x_j] \text{ if } \{x_i, x_j\} \in \Gamma^1 \rangle$$

Viceversa a group with a presentation of this kind uniquely determines a simplicial graph in the obvious way.

Example 3.3.2. A discrete graph on n points gives rise to F_n, the free group on n generators, whereas a complete one (i.e. each couple of vertices is joined by an edge) yields the free abelian group $\mathbb{Z}^n$. RAAGs can be thought as a kind of "interpolation" between these two extreme cases.

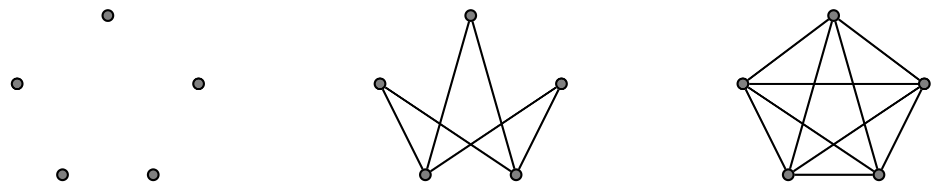

Figure 3.8: Three graphs, giving rise respectively to to F_5, $F_2 \times F_3$ and $\mathbb{Z}^5$

The presentation complex $X_{A(\Gamma)}$ of a RAAG $A(\Gamma)$ has a particularly nice description in terms of the underlying graph Γ. First of all, loops in $X_{A(\Gamma)}$ correspond to points in Γ. Then, since all the relations appearing in the standard presentation of $A(\Gamma)$ are commutators, each 2-cell is glued in the shape of a torus $S^1 \times S^1$; in other words we take a square and impose the identification $x_i x_j x_i^{-1} x_j^{-1}$ on its boundary for some couple of commuting generators x_i, x_j of $A(\Gamma)$, that is for some couple of adjacent vertices x_i, x_j of Γ. Therefore $X_{A(\Gamma)}$ is built from a bunch of tori glued together along their principal parallels and meridians.

For example, the following picture shows how to construct the complex for $A(\bullet\!-\!\overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\bullet)$.

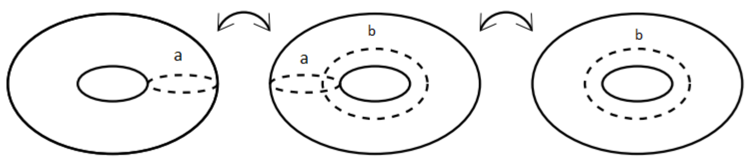

Figure 3.9: The presentation complex for $A(\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet)$

Remark 3.3.3. As one may expect, it is possible to add higher dimensional cells to $X_{A(\Gamma)}$ to obtain a $K(A(\Gamma), 1)$. The standard construction found in the literature about RAAGs just glues an n-dimensional torus for each set of n pairwise commuting generators of $A(\Gamma)$. This operation "fills the holes" and kills higher dimensional homotopy. The resulting complex Σ_Γ is known as the Salvetti complex of $A(\Gamma)$ and turns out to be indeed a $K(A(\Gamma), 1)$, but we will not need this in the following. The complex in the picture above is actually the Salvetti complex because in that example there are no 3-dimensional relations, so one has not to add higher dimensional tori to the presentation complex. For further properties of the Salvetti complex also have a look at 3.4.5 below.

3.3.2 A-typing of a Square Complex

To go from a cube complex to a RAAG we need a simplicial graph. For our purposes it will be enough to consider square complexes, i.e. 2-dimensional cube complexes.

Definition 3.3.4. Let B a square complex. We define the hyperplane graph (or intersection graph) Γ_B as follows: take a vertex for each hyperplane of B and join two vertices if and only if the corresponding hyperplanes intersect.

Remark 3.3.5. We observe that

- By construction Γ_B has no bigon.
- If B has no selfintersecting hyperplanes, then Γ_B has no loops.

In this case Γ_B is a simplicial graph and we can construct $A(\Gamma_B)$. We can think of this group as presented by

$$A(\Gamma_B) = \langle x_i \mid [x_i, x_j] \text{ iff } x_i \cap x_j \neq \varnothing \rangle$$

where the x_i denote either the hyperplanes or the walls of B. Let $X_B := X_{A(\Gamma_B)}$ the presentation complex associated to this presentation of $A(\Gamma_B)$. We can be even more explicit: loops in X_B correspond to hyperplanes (or walls) in B and there is a 2-cell (glued as a torus as describer before) for each couple of intersecting hyperplanes (or walls).

Lemma 3.3.6. *If hyperplanes in B are embedded and 2-sided then we can find a combinatorial map $\tau_A : B \to X_B$.*

Proof. By hypothesis, we can coherently orient each wall of B. So it is possible to define a map f on the 1-skeleton which sends each edge a of B to the loop in X_B associated to the wall through a in a way that preserves orientations. But then such a map is easily extended to squares of B in the following way. Take a square S of B and let W_i, W_j the walls containing its edges; they are distinct walls since B has embedded hyperplanes. The perimeter of S is mapped by f on the loop $x_i x_j x_i^{-1} x_j^{-1}$ where x_i, x_j are the loops corresponding to the walls W_i, W_j. Since W_i, W_j intersect in S there is a square (a torus) in X_B with boundary this loop, and it is unique by construction of Γ_B. Then we just send S to this torus. □

Definition 3.3.7. If B is a square complex with embedded and 2-sided hyperplanes, we say that the map $\tau_A : B \to X_B$ in the previous lemma is an A-typing of B. In the following, if we talk about A-typings we implicitly suppose that B is a square complex with embedded and 2-sided hyperplanes so that everything is well-defined.

Now we want to show that if moreover B is A-special, then any A-typing is π_1-injective. This will follow from a close inspection of elementary homotopies in X_B and the fact that A-speciality will allow us to pullback such homotopies.

We recall from 3.2.9 that, by cellular approximation, we can reduce the study of homotopies of paths in a cube complex to the consideration of elementary homotopies of an edge-path $x_1 \dots x_n$, which can only take one of these two forms:

1. $x_1 \dots x_{i-1} x_i x_i^{-1} x_{i+1} \dots x_n \simeq x_1 \dots x_{i-1} x_{i+1} \dots x_n$

2. $x_1 \dots x_i x_j \dots x_n \simeq x_1 \dots x_h x_k \dots x_n$ whenever $x_i x_j x_k^{-1} x_h^{-1}$ is the (oriented) perimeter of some square of X

In the case of X_B, the homotopy of type 2 takes the particular form

2. $x_1 \dots x_i x_j \dots x_n \simeq x_1 \dots x_j x_i \dots x_n$ whenever the hyperplanes x_i and x_j intersect in B

This is just because squares are glued along commutators of edges if and only if the corresponding hyperplanes intersect. Here we are using the symbols x_i to denote loops in X_B and also the corresponding walls or hyperplanes in B.

Theorem 3.3.8. *Let B be an A-special square complex. Then any A-typing $\tau_A : B \to X_B$ induces an embedding $\tau_{A_*} : \pi_1(B) \hookrightarrow \pi_1(X_B) = A(\Gamma_B)$.*

Proof. Let γ be an edge-loop in B and let $\gamma_* = \tau_A(\gamma)$ be its image in X_B. First observe that γ and γ_* have the same combinatorial length. Suppose γ_* is nullhomotopic in X_B. This means that there is a sequence of homotopies from γ_* to the unique constant path in the base point of X_B. We have to consider what happens to γ in B when γ_* undergoes a homotopy in X_B.

1. If in X_B we see a homotopy of the first type, i.e. $x_i x_i^{-1} \simeq \varnothing$, it means that γ has two consecutive edges from the same wall, but the second with opposite orientation. If these two edges were distinct, then they would realize a configuration of self-intersection (if they lied in some square) or of direct osculation (if not); but both cases would be absurd since B is A-special. Thus these two edges are actually the same edge (traversed with different orientations), then we have that γ has a piece of the form aa^{-1} which can be homotoped to a point.

2. If in X_B we see a homotopy of the second type, i.e. $x_i x_j \simeq x_j x_i$, it means that γ has two consecutive edges from two different walls, but these walls intersect somewhere in B. The case in which they do not lie in a square is absurd because it would give a configuration of interosculation in B, which is A-special. So these edges lie in a square, then their concatenation can be homotoped to the concatenation of the two opposite edges (preserving orientation).

Therefore we see that every homotopy of γ_* gives a homotopy of γ of the same kind. In particular, if γ_* undergoes a homotopy which reduces its length by 2, then γ undergoes a homotopy which reduces its length by 2 as well[1]. This allows us to conclude that if γ_* is nullhomotopic then γ is nullhomotopic. As a consequence the A-typing τ_A is π_1-injective. □

Remark 3.3.9. We explicitly observe that the square complex X_B considered in the above construction and proof is actually the 2-skeleton of the Salvetti complex Σ_{Γ_B} of the RAAG $A(\Gamma_B)$ associated to the hyperplane graph Γ_B of B. The Salvetti complex of a RAAG was introduced in 3.3.3, and more about it will be said in section 3.4, where it will be showed that it carries an interesting geometry of non-positive curvature.

[1] Notice that this proof also shows that nullhomotopic edge-loops have even length.

As an application of this theorem, we get the following remarkable result which applies to a cube complex of arbitrary dimension. Here (and in the following) something holds *virtually* for a group if it holds for a finite index subgroup, and for a topological space if it hold for a finite covering space.

Corollary 3.3.10. *Let C be a compact special cube complex. Then $\pi_1(C)$ virtually embeds in a finitely generated RAAG.*

Proof. By Theorem 3.2.11 we can find a finite cover $B \to C$ which is A-special. Equivalently, B^2 is A-special; since it is a square complex, the previous theorem applies and gives an embedding

$$\pi_1(B) = \pi_(B^2) \hookrightarrow \pi_1(X_{B^2}) = A(\Gamma_{B^2})$$

so $\pi_1(C)$ has a finite index subgroup $\pi_1(B)$ which embeds in a RAAG; this RAAG is finitely generated because compactness of C implies that B^2 has a finite number of hyperplanes. □

3.4 Non-Positively Curved Cube Complexes

As remarked in 3.2.6, it is possible to turn a cube complex into a metric space. Even if some properties of cube complexes may be established by direct arguments (as seen above for their virtual embedding in RAAGs), the combinatorial theory is deeply intertwined with the resulting geometric properties. This is quite beautiful on its own, provides an interplay between combinatorial and geometric aspects and moreover is the basis for the original proof of the virtual embedding given in [HaW08]. In this section we introduce the main ideas and result concerning this point of view.

By definition, a cube complex X is obtained by glueing euclidean cubes $\{C_i\}$ via isometries of their faces, therefore the various euclidean metrics $\{d_i\}$ defined on each cell match together to give a global metric d_X on the whole cube complex; what is important here is that all the edges insisting on a vertex have the same length. This is an instance of a general construction for M_k-polyhedral complexes, see chapter I.7 in [BrH99], especially I.7.10 and I.7.32, which prove that the metric space obtained in this way is a geodesic metric space which is lenght and complete (if finite dimensional). The very nice thing about this metric space is that many of its geometric features are encoded in the underlying combinatorics.

Definition 3.4.1. A flag complex is a simplicial complex in which $\{k+1\}$ pairwise adjacent vertices always span a k-simplex.

In other words a flag complex is determined by its 1-skeleton: every time you see the 1-skeleton of a k-simplex, there actually is a k-simplex, no holes allowed. Recall from 3.1.9 that a cube complex is called simple when the link of each vertex is simplicial complex (and that all the cube complex we consider are simple).

Definition 3.4.2. We say a cube complex X is NPC if the link of each vertex is a flag complex.

Example 3.4.3. The link of a vertex in I^3 is a 2-simplex, which is of course a flag complex, thus a cube is NPC; on the other hand the link of a vertex in ∂I^3 is the complete graph on 3 vertices, which is not flag. Notice that the geometric realization of ∂I^3 is homeomorphic to a sphere.

The acronym NPC stands for non-positively curved. As one may expect this is because the metric space (X, d_X) defined above is non-positively curved in the sense of the CAT(k) condition (1.2.4). This is actually the case, as Gromov pointed out in his seminal paper on hyperbolic groups [Gro87].

Theorem 3.4.4 (Gromov Link Condition)**.** *A finite dimensional cube complex has non-positive curvature if and only if it is NPC.*

For a self contained discussion of this result see [BrH99], chapter II.5, where this is Theorem 5.20. Moreover a simply connected NPC cube complex is also called a CAT(0) cube complex, since it is actually a CAT(0) metric space.

Remark 3.4.5. In 3.3.3 we introduced the Salvetti complex Σ_Γ of a RAAG $A(\Gamma)$. It directly follows from its construction that each link is a flag complex, thus Σ_Γ is an NPC cube complex. As a result its universal cover is a CAT(0) cube complex; it is contractible by 1.2.7, which implies that the Salvetti complex Σ_Γ is a $K(A(\Gamma), 1)$.

The correspondence between combinatorial and geometric properties also holds for maps: a local isometry in the combinatorial sense of 3.2.5 turns out to be a local isometry in the metric sense. Moreover one has the following.

Proposition 3.4.6. *Any A-typing of an A-special square complex is a local isometry into the* 2*-skeleton of the Salvetti complex of the associated hyperplane graph.*

The idea of the proof of this is essentially the same idea behind our investigation of elementary homotopies in 3.3.8. This fact, combined with 3.4.5 and 1.2.8 gives another proof of the virtual embedding in a RAAG. However we remark that a precise proof of this involves some quite technical steps. For instance one has to show that a special cube complex is NPC, which in turns needs the fact that a NPC square complex admits a canonical NPC cube completion such that any map defined on the square complex extends to the whole completion. These issues are addressed by Haglund and Wise in the appendix of [HaW08].

Chapter 4

Decompositions of 3-Manifolds

The topology of a surface can be studied cutting it along embedded curves (1-codimensional submanifolds) and thus producing simpler pieces; in a similar fashion 3-dimensional manifolds can be cut along embedded surfaces (which are the 1-codimensional objects here).

In this chapter we introduce the study of 3-dimensional manifolds from the point of view of their decomposition along surfaces. This approach will provide us with some useful reduction as well as some deep ideas. The contents are organized according to the increasing complexity of the surfaces we want to cut along.

The 3-manifolds in this chapter are always assumed to be compact, connected and orientable. If S is a surface in M we denote by $M \setminus\setminus S$ the compact (but possibly disconnected) submanifold of M obtained by removing an open tubular neighborhood of S. We will also say that S is separating (respectively, non-separating) if $M \setminus\setminus S$ is disconnected (respectively, connected).

First of all we recall the fundamental theorem of low-dimensional topology, which will be implicitly used in the following.

Theorem (Moise)**.** *Every topological manifold of dimension $n \leq 3$ can be given a differentiable structure and a piecewise linear structure in a unique way, up to isomorphism in the respective category.*

A standard reference for results concerning smoothings and triangulations of topological manifolds is [KS77].

4.1 Prime Decomposition

The simplest kind of surface one can find in a 3-manifold is the sphere S^2, so we begin with the study of spheres in 3-manifolds. Standard references for the material in this section are [Hat] and [Hem76]. Both of them date back before Perelman's proof of Poincaré Conjecture[1]. In this section we try, where possible, to give shorter proofs of classic result by exploiting this new strong tool.

Definition 4.1.1. An embedded sphere S in a 3-manifold M is essential if it does not bound a ball and if it is not a boundary component. M is said to be irreducible if it does not contain an essential sphere; in other words every sphere in M either bounds a ball or is a boundary component.

Remark 4.1.2. If a sphere bounds a ball then it is of course homotopically trivial, i.e. it can be continuously contracted to a point in M. The celebrated Sphere Theorem by Papakyriakopoulos (see Theorem 4.3 in [Hem76]) roughly states that every non trivial class in $\pi_2(M)$ gives rise to an embedded sphere, giving a strong converse to the previous statement. This means that M is irreducible if and only if $\pi_2(M) = 1$, which translates a geometric condition into a homotopy-theoretic one.

Example 4.1.3. By a classic result of Alexander[2], every sphere in $\mathbb{R}^3$ bounds a ball. Therefore $\mathbb{R}^3$ is a (non compact) irreducible manifold, which is of course the local model for any (compact or not) manifold. In other words every interior point in a 3-manifold sits inside a ball whose boundary is an inessential sphere. Building on the same result one can prove that S^3 is irreducible as well: every embedded sphere bounds a ball (on both sides).

Example 4.1.4. Consider the manifold $S^2 \times I$. The spheres $S^2 \times \{-1\}$ and $S^2 \times \{1\}$ are not essential, simply because they are boundary components. But the sphere $S^2 \times \{0\}$ is essential. If we glue the boundary components together we get the product $S^2 \times S^1$ (if we use an orientation reversing homeomorphism of S^1) or the twisted (non trivial) bundle $S^2 \widetilde{\times} S^1$ (if we

[1]The Poincaré Conjecture claims that the only closed orientable simply-conncected 3-manifold is S^3. Perelman proved this is true in a series of papers on the arXiv in 2002-3.

[2]A detailed proof can be found in Theorem 1.1 of [Hat].

choose an orientation preserving omeomorphism of S^1); in both cases the fiber F of the bundle is an essential sphere. Since we are working with orientable manifolds, we will forget about the second one.

We are interested in understanding what happens when we cut along a sphere S (which is not a boundary component) in a 3-manifold M. If the sphere S is not essential, then it bounds a ball and thus disconnects M. As a result $M \setminus \setminus S = M' \sqcup B^3$, where M' is a 3-manifold with (at least) one spherical boundary component and B^3 is a closed 3-ball. This is just the same as representing M as $M \# S^3$, which are exactly the manifolds we obtain if we cap off the spherical boundaries resulting from the cutting procedure.

Motivated by this simple case, we turn our attention to the operation of connected sum. To get a first taste of how decomposing in a connected sum reduces the topological complexity of the manifolds involved, we consider the following lemma, which is a standard application of Seifert-van Kampen and the fact that S^2 is simply connected.

Lemma 4.1.5. *If $M = M_1 \# M_2$ then $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$.*

We are now ready for the following definition.

Definition 4.1.6. A 3-manifold is said to be prime if every time it can be written as a connected sum $M = M_1 \# M_2$ then we have that either $M_1 = S^3$ or $M_2 = S^3$.

This definition is exactly the same as the previous definition of irreducible, modulo one of the examples discussed above.

Proposition 4.1.7. *M is prime if and only if it is either irreducible or homeomorphic to $S^2 \times S^1$.*

Proof. Suppose M is prime and let S be an embedded sphere in M. We have two cases here:

1. If S is separating then let M_1 and M_2 denote the two components of $M \setminus \setminus S$, which have $\partial M_i \cong S$. Let $\widehat{M_i}$ be the manifold obtained capping ∂M_i with a 3-ball. Then $M = \widehat{M_1} \# \widehat{M_2}$. Since M is prime we have that one of the connected summands is just S^3; let's say $\widehat{M_1} = S^3$; but this implies $M_1 = B^3$, so S bounds a ball and M is irreducible.

2. If S is non-separating, then we can find a curve γ in M intersecting S in a single point trasversely and a tubular neighborhood R of $S \cup \gamma$. Then we have that R is homeomorphic to $S^2 \times S^1$ with a ball removed and that ∂R is a sphere which disconnects M. By the discussion of the previous case we have that $M = (S^2 \times S^1)\#N$ for some 3-manifold N. But since M is prime we have $N = S^3$ and $M = S^2 \times S^1$.

For the reverse implication, we show that an irreducible manifold is necessarily prime and that the same holds for $S^2 \times S^1$ even if it is not irreducible.

1. Suppose M is irreducible and $M = M_1\#M_2$. Then M is obtained from $N_1 = M_1 \setminus B^3$ and $N_2 = M_2 \setminus B^3$ by glueing them along the boundary sphere; this sphere bounds both of them in M and by irreducibility one of them, say N_1, is just a ball, which means that $M_1 = S^3$.

2. Now suppose $S^2 \times S^1 = U\#V$. By 4.1.5 we have $\mathbb{Z} \cong \pi(S^2 \times S^1) \cong \pi_1(U) * \pi_1(V)$, and so we get that one of the summands, say U, is simply connected. From the definition of connected sum, we have a 2-sphere $S \subset S^2 \times S^1$ such that $(S^2 \times S^1) \setminus \setminus S$ has two components U' and V' which are the compact manifolds with boundary a sphere obtained by removing a 3-ball from U and V respectively. Since S is simply connected, by Seifert-van Kampen we have that $\pi_1(U) = \pi_1(U')$ (and also $\pi_1(V) = \pi_1(V')$); in particular U' is simply connected. The universal cover of $S^2 \times S^1$ is $S^2 \times \mathbb{R} \cong \mathbb{R}^3 \setminus \{0\}$, and U' lifts to an homeomorphic copy $\widetilde{U'}$ of itself here. Since $\partial\widetilde{U'}$ is a sphere in $\mathbb{R}^3$, it bounds a ball by 4.1.3, so $\widetilde{U'}$ is a ball and then U' is a ball too. This implies that $U \cong S^3$ and thus $S^2 \times S^1$ is prime.

$\square$

It turns out that connected sum is the right thing to look at when cutting along embedded spheres, as the following result shows.

Theorem 4.1.8 (Prime Decomposition - Kneser 1929, Milnor 1962)**.** *Each 3-manifold M can be decomposed as a connected sum of a finite number of prime 3-manifolds $M = P_1\#\ldots\#P_n$. This decomposition is unique up to the order of the factors and up to insertion or deletion of S^3's factors.*

The classical proof is combinatorial and can be found in [Hat] or [Hem76]: it is straightforward but quite technical. Here we give a short proof of the existence of the decomposition in the closed case, which nevertheless relies on the Poincaré Conjecture.

Proof of Theorem 4.1.8. If M has no essential spheres, then it is irreducible and so prime, by 4.1.7. Otherwise M will contain some essential spheres. From the proof of 4.1.7 it follows that splitting S along a non-separating sphere allows to decompose M as a connected sum $M = M'\#(S^2 \times S^1)$ for a suitable M', i.e. we can pull out a prime factor. Then iterating this we are able to decompose M as $M = N\#(S^2 \times S^1)\# \dots \#(S^2 \times S^1)$, where N is a 3-manifold in which every essential sphere is separating.

Now we keep on splitting N along such separating spheres until we are able to find them and we obtain $M = \underset{i\in I}{\#} P_i$, where P_i is prime and closed (since M was) and I possibly infinite. By 4.1.5 we get $\pi_1(M) = \underset{i\in I}{*} \pi_1(P_i)$. But since fundamental groups of manifolds are finitely generated, we have that $\pi_1(P_i) = 1$ for almost all $i \in I$, and by Poincaré Conjecture this means that $P_i \cong S^3$ for almost all $i \in I$. The thesis follows since $N\#S^3 = N$ for every 3-manifold N. □

Since we have seen that a prime factor is either homeomoprhic to $S^2 \times S^1$ or irreducible, this theorem essentially reduces the study of 3-manifolds to that of irreducible 3-manifolds.

4.2 Incompressible Surfaces

In the previous section we have described the theory of cutting along spheres, and this has required some "ad hoc" techniques. Since the theory for higher genus surfaces is more uniform, in this section we gather the main definitions and results that will be used in the following ones. Throughout this section M will denote a compact, connected and orientable 3-manifold, and S a compact orientable surface embedded in M, with no sphere components and not necessarily connected.

Definition 4.2.1. When S is connected, we say that S is 2-sided (respectively 1-sided) if the normal bundle to S is trivial (respectively non trivial). This definition can be extended to a non connected surface such that all of its components are 2-sided (or all are 1-sided).

Remark 4.2.2. One may expect that 2-sidedness is equivalent to orientability; this holds if M is orientable. But in full generality the two properties are not the same; we can find examples of each of the four possible combinations:

1. $S^2 \subset S^2 \times S^1$ is orientable and 2-sided;
2. $S^1 \times S^1 \subset \mathbb{RP}^2 \times S^1$ is orientable but 1-sided;
3. $\mathbb{RP}^2 \subset \mathbb{RP}^2 \times S^1$ is non orientable but 2-sided;
4. $\mathbb{RP}^2 \subset \mathbb{RP}^3$ is non orientable and 1-sided.

Definition 4.2.3. A surface $S \subset M$ without disk components is incompressible if it is properly embedded (i.e. embedded so that $\partial S = S \cap \partial M$) and if for every disk $D \subset M$ with $D \cap S = \partial D$ there is another disk $D' \subset S$ such that $\partial D = \partial D'$.

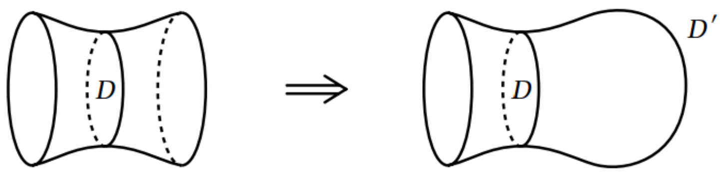

Figure 4.1: An incompressible surface

The idea is that if S is incompressible and we cut along ∂D and cap off the resulting boundaries with disks, we are not actually simplifying its topology, but we are just splitting off a sphere.

Example 4.2.4. Each surface (different from a union of spheres) in $\mathbb{R}^3$ or S^3 is compressible.

The following lemma gives a homotopy-theoretic translation of this geometric definition.

Lemma 4.2.5. *Let $S \subset M$ be a 2-sided surface*[3]*. Then S is incompressible if and only if it is π_1-injective, i.e. $\pi_1(S) \hookrightarrow \pi_1(M)$.*

Proof. Let S be incompressible and suppose $\pi_1(S) \to \pi_1(M)$ is not 1-1. Let γ be a loop which is nullhomotopic in M but not in S. Let $f : D^2 \to M$ the homotopy (in M) between γ and a point. By Stalling's Loop Theorem (see Theorem 4.2 in [Hem76]) we can choose f to be injective, so that γ bounds a disk in M. Since it is not nullhomotopic in S it does not bound a disk in S, which is absurd by incompressibility assumption.

Suppose now S is π_1-injective and let $D \subset M$ a disk with $D \cap S = \partial D$. Then ∂D represents a class in $\pi_1(S) \hookrightarrow \pi_1(M)$, and is trivial in $\pi_1(M)$ since here it bounds D. But by injectivity it also represents the trivial class in $\pi_1(S)$. But a nullhomotopic loop on a surface bounds a disk, so we see S is incompressible. $\square$

Example 4.2.6. Here is an example of incompressible surface. This is much more interesting than the previous one and will concern us also during the next chapters. Let M be a surface bundle over S^1, with fiber some fixed closed orientable surface[4] S of genus $g \geq 1$. From the long exact sequence of homotopy groups associated to this fibration we extract the following piece

$$0 = \pi_2(S^1) \to \pi_1(S) \to \pi_1(M)$$

from which we get that S is π_1-injective and thus incompressible by 4.2.5. Since the base space of this kind of bundle is just S^1, the topological structure is quite easy to understand. Removing a point from the base is equivalent to removing a fiber surface from the bundle: this produces a product manifold $S \times [0, 1]$. This means that M is a mapping torus for some homeomorphism of S to itself; in other words it can be obtained as

$$M = {}^{S \times [0,1]}\big/_{(p,0) \sim (\varphi(p),1)}$$

for some $\varphi \in Homeo(S)$.

[3] Since our M is always orientable, by the previous remark this is the same as saying "let $S \subset M$ be an orientable surface".

[4] In Chapter 5, we will say that in this case M fibers over the circle, or just that M is fibered.

4.3 JSJ Decomposition

Once we have established the Prime Decomposition we can reduce the study of (compact oriented) 3-manifolds to the study of their prime factors. Among these we have some $S^2 \times S^1$'s and the generic factors are irreducible. So the natural thing to do now is to take irreducible manifolds and try to cut along embedded tori.

Definition 4.3.1. An irreducible manifold M is atoroidal if every incompressible torus is isotopic to some component of ∂M.

In some intuitive sense, atoroidal manifolds are to incompressible tori what irreducible manifolds are to essential spheres. The expectation (motivated from the analogy with the Prime Decomposition) is that we can cut along incompressible tori until we are left with atoroidal pieces. Notice anyway that here we are not capping the resulting boundaries as we did in the Prime Decomposition. This is due to the fact that if you want to fill a spherical boundary component by glueing a ball, you have to choose an automorphism of the boundary sphere, and there is essentially only one choice (up to orientation and isotopy): the mapping class group of the sphere is trivial. On the other hand the torus has a non trivial mapping class group, so there is no canonical way of capping the boundaries.

In the Prime Decomposition we do not actually cut until we have only irreducible pieces: when we meet some exceptional (but nice enough) piece (i.e. $S^2 \times S^1$) we do not cut it along one of its essential surfaces, but content ourselves with a decomposition in prime factors. In an analogous way, here we get some exceptional pieces which are nice enough to deserve not to be split. These pieces are the so-called Seifert manifolds (see next paragraph).

Theorem 4.3.2 (JSJ Decomposition - Jaco, Shalen, Johanson, 1978)**.** *Every irreducible 3-manifold M admits a finite collection of disjoint incompressible tori $T_1, \dots, T_n$ such that the connected components of $M \setminus \setminus (T_1 \cup \dots \cup T_n)$ are either atoroidal or Seifert manifolds. A minimal such collection is unique up to isotopy.*

Remark 4.3.3. Before introducing Seifert manifolds, we observe that the exceptional piece $S^2 \times S^1$ contains no incompressible torus: since a torus in

$S^2 \times S^1$ is necessarily 2-sided, this follows from 4.2.5 and the fact that $\pi_1(S^2 \times S^1) = \mathbb{Z}$ cannot contain the fundamental group of a torus. As a consequence the theorem holds for a general prime (always compact, connected and orientable) 3-manifold, since $S^2 \times S^1$ is itself atoroidal.

4.3.1 Seifert Manifolds

Now we describe the basic features of these exceptional pieces in the JSJ Decomposition.

Definition 4.3.4. Let $p, q \in \mathbb{N}$ be coprime integers. We call a $(p,q)-fibered$ torus the manifold obtained from a solid cylinder $B^2 \times I$ by glueing the two boundary components after a rotation of angle $2\pi\frac{p}{q}$. This is a solid torus $B^2 \times S^1$ with a foliation in circles coming from the foliation of the cylinder by straight segments $\{p\} \times I$.

Definition 4.3.5. A Seifert manifold is a 3-manifold M with a decomposition into disjoint circles such that each point has a neighbourhood which is isomorphic (preserving fibers) to a $(p,q) - fibered$ torus (for some $p, q \in \mathbb{N}$).

Definition 4.3.6. Let C be a fiber in a Seifert manifold. We say that C has multiplicity q if there is a small disk transverse to C such that if a fiber intersects C then it intersects it in q distinct points. Then C is simple (or regular) if its multiplicity is 1, and exceptional (or singular) otherwise.

Example 4.3.7. In a $(p,q) - fibered$ torus the central fiber $\{0\} \times S^1$ has multiplicity q and any other fiber is simple. Since this gives the local model for every Seifert manifold, we see that in an arbitrary Seifert manifold exceptional fibers are isolated.

Remark 4.3.8. As a consequence of the previous example we have that if we identify each fiber to a point, the quotient is homeomorphic to a surface S; a point on S is said to be of multiplicity q if it is the equivalence class of a fiber of multiplicity q. This surface actually turns out to admit the structure of a 2-dimensional orbifold with cone points singularities: the stabilizer of each point is exactly $\mathbb{Z}/_{q\mathbb{Z}}$, where q is the multiplicity of the point.

Moreover the projection $p : M \to S$ is an ordinary fiber bundle with fiber

S^1 in the complement of the exceptional points. If one likes the orbifold machinery, it makes sense to call p an orbifold bundle.

The reason why we are dealing with this point of view is that Seifert manifold have been explicitly classified in terms of (invariants of) the base orbifold and the bundle projection. Lens spaces and S^1-bundles over orientable compact surfaces belong to this classification.

4.4 Geometric Decomposition

The previous decompositions deal with homotopy-theoretic properties of the manifold: once you perform a Prime Decomposition on your manifold, every sphere you find will be not essential, and the same can be said about tori in a JSJ Decomposition (up to some exceptions in both cases, as we have seen). In this section we want to focus on a decomposition which has a strong geometric flavour; the decomposition process is quite close to the JSJ one, but somehow different, as we will see in the end.

It is a classic and well known result that every closed connected orientable surface admits a geometry (elliptic, euclidean or hyperbolic) according to its topology (genus $0, 1$ or > 1). In [Thu82] Thurston conjectured that the picture was similar (but different in some crucial features) for the 3-dimensional world. The main difference with respect to the case of surfaces is first of all that one cannot expect an arbitrary 3-manifold to admit any "nice geometry", and then that one has to redefine what a "nice geometry" is, because there are 3-dimensional geometries which are not equivalent to geometries of constant curvature.

The intuitions of Thurston have guided the research in the field of 3-dimensional topology for some decades. Only recently (2002-3) Perelman has provided a proof of Thurston's Conjecture. The aim of this section is to give a description of this result and of the concepts involved.

Theorem 4.4.1 (Thurston Geometrisation Conjecture, 1982 - *now* Perelman Theorem, 2003)**.** *An irreducible 3-manifold M admits a finite collection of disjoint incompressible tori $T_1, \dots, T_n$ such that the interior of each connected component of $M \setminus\setminus (T_1 \cup \cdots \cup T_n)$ can be endowed with a geometric structure* *of finite volume.*

In the next paragraphs we clarify this statement and give a lot of examples.

4.4.1 Geometric Structures on Manifolds

Here we want to define a geometry on a manifold; this is essentially done specifying which are the allowed transformation. In order to give a precise meaning to the term "geometric structure", we will need some technical definitions.

Definition 4.4.2. Let X be a topological space. A pseudogroup $\mathcal{G}$ on X is a collection of local homeomorphisms between open sets of X such that

- the domain of the elements of $\mathcal{G}$ covers X,
- G is closed under restriction, composition and inversion, when these operations are defined,
- being in $\mathcal{G}$ is a local property, i.e. if $U = \cup U_i$ and $f : U \to V$ is some local homeomorphism such that $f|_{U_i} \in \mathcal{G}$ then $f \in \mathcal{G}$.

Example 4.4.3. Obvious examples are given by the pseudogroups $\mathcal{C}^k$ of C^k local diffeomorphisms between open sets in $\mathbb{R}^n$ for $k = 0, \ldots, \infty$ and the pseudogroup $\mathcal{H}$ of holomorphic maps between open sets in $\mathbb{C}^n$. Another example is given by the pseudogroup $\mathcal{PL}$ of local piecewise-linear homeomoprhisms between open sets in $\mathbb{R}^n$.

Example 4.4.4. More generally, if G is a Lie group acting on some manifold X, it gives rise to a pseudogroup, still denoted G, on X generated by the restrictions of elements of G to open sets in X. The most interesting example is given by the isometry group of a Riemannian manifold.

Following the classical definition of a differentiable manifold, we give the following one.

Definition 4.4.5. Let X be a manifold, $\mathcal{G}$ be a pseudogroup on X and M a topological space.

- A $(\mathcal{G}, X)$-chart on M is an open set U of M together with a homeomorphism $\varphi_U : U \to \varphi_U(U) \subset X$. Two $(\mathcal{G}, X)$-charts (U, φ_U) and (V, φ_V) are compatible if the transition map $\varphi_U \varphi_V^{-1} : \varphi_V(U \cap V) \to \varphi_U(U \cap V)$ belongs to $\mathcal{G}$.

- M is a $(\mathcal{G}, X)$-manifold if it is Hausdorff, second countable and endowed with a maximal collection of compatible charts; we also say that M has a geometric structure of type $(\mathcal{G}, X)$.

Example 4.4.6. A C^k-manifold is just a $(\mathcal{C}^k, \mathbb{R}^n)$-manifold, a complex manifold is just a $(\mathcal{H}, \mathbb{C}^n)$-manifold and a PL-manifold is just a $(\mathcal{PL}, \mathbb{R}^n)$-manifold.

This approach allows to treat on a common ground a great variety of geometries that one can put on manifolds. The idea is that we start with some model geometry embodied by some manifold X and select some admissible transformation which belong to that geometry (the pseudogroup) and use these to construct new manifolds glueing together pieces of X through these geometry-preserving transformations.

Definition 4.4.7. A pair (G, X) is called a model geometry if

1. X is a connected and simply connected differentiable manifold;

2. G is a Lie group of diffeomorphisms of X acting transitively with compact stabilizers, and is maximal with respect to this property;

3. there exists at least one compact (G, X)-manifold.

Proposition 4.4.8. *Every model geometry (G, X) can actually be endowed with a G-invariant Riemannian metric; therefore G can actually be thought of as a group of Riemannian isometries.*

Proof. Let $x \in X$ and choose some inner product $(.,.)$ on T_xX. Since stabilizers are compact, we can average with respect to Haar measure on the stabilizer in x and define

$$< u, v >= \int_{G_x} (dg_x u, dg_x v) dg$$

This turns out to be a G-invariant inner product on T_xX and we get a Riemannian metric by translating it with the action of G, which is transitive by hypothesis. $\square$

4.4.2 Thurston's Eight Geometries

Thurston has classified all the model geometries that can occur in dimension 2 and 3. As one may expect, in dimension 2 we find only $\mathbb{S}^2$, $\mathbb{E}^2$ and $\mathbb{H}^2$ with their groups of isometries. Of course also in dimension 3 we have $\mathbb{S}^3$, $\mathbb{E}^3$ and $\mathbb{H}^3$ with their groups of isometries: these are the so-called homogeneous geometries and have stabilizers isomorphic to $O(3)$. Compact representative are respectively given by $\mathbb{S}^3$, the 3-torus $S^1 \times S^1 \times S^1$ and the manifold obtained from $\mathbb{S}^3$ removing an open tubular neighbourhood of a trefoil or figure-eight knot.

But, has hinted above, the third dimension leaves enough room for the existence of non homogeneous model geometries, which have typically smaller stabilizers. We now give a brief description of some geometries of this kind.

Example 4.4.9. The manifold $\mathbb{S}^2 \times \mathbb{R}$ can be endowed with the product metric; the isometry group of this structure is $Isom(S^2) \times \mathbb{R} \cong O(3) \times \mathbb{R}$ with stabilizers isomorphic to $O(2)$. A compact manifold with this kind of geometry is given by $S^2 \times S^1$.

Example 4.4.10. The same construction can be carried on for $\mathbb{H}^2 \times \mathbb{R}$: we get an isometry group isomorphic to $PSL_2\mathbb{R} \times \mathbb{R}$ with stabilizers again isomorphic to $O(2)$. Any product $S \times S^1$ where S is a hyperbolic surface is a compact example of this geometry.

Of course the same construction with $\mathbb{E}^2$ gives the geometry $\mathbb{E}^3$. There are three more model geometries, which are quite more exotic.

Example 4.4.11. Let $\widetilde{SL_2\mathbb{R}}$ be the universal cover of $SL_2\mathbb{R}$. It is possible to induce a metric on this Lie group from the metric of the hyperbolic plane $\mathbb{H}^2$. The bundle of unit tangent vectors on a closed hyperbolic surface is a compact representative of this geometry.

Example 4.4.12. Let Nil be the Heisenberg group of upper triangular matrices with real coefficients and unit diagonal. This Lie group is diffeomorphic to $\mathbb{R}^3$. We impose the euclidean inner product at the identity and then propagate via group multiplication. This gives an invariant metric on *Nil*. A compact example is obtained by *Nil* itself quotienting out the lattice of matrices with integer coefficients. Manifolds with this geometry are called nilmanifolds.

Example 4.4.13. Consider $\mathbb{R}^3$ with the group structure given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) := (x_1 + \exp^{-z_1} x_2, y_1 + \exp^{z_1} y_2, z_1 + z_2)$$

As above we construct a left-invariant metric on this Lie group, and we call *Sol* the resulting model geometry and solvmanifold the manifolds modelled on it. A compact example can be built as follows: let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ the linear mapping defined by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Since it fixes the integer lattice, it descends to a diffeomorphism (of Anosov type) of the torus $T = S^1 \times S^1$. The mapping torus of this diffeomorphism is an example of solvmanifold.

As announced, we have the following theorem of Thurston.

Theorem 4.4.14 (Thurston)**.** *A 3-dimensional model geometry (G, X) is one of the eight geometries discussed above.*

Proof. (Sketch, see Theorem 3.15 of [Wil02] for a detailed proof) Since the metric constructed in 4.4.8 is G-invariant, the stabilizer of a point is a (maximal connected) subgroup of $O(3)$, so it must be $SO(3)$, $SO(2)$ or the trivial group. Now the proof boils down to the inspection of these cases. The first case is that of homogeneous geometry of constant curvature ($\mathbb{S}^3$, $\mathbb{E}^3$ and $\mathbb{H}^3$). The second case has 2-dimensional stabilizer and the third dimension is left free, allowing for "foliated" geometries: here we find $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, *Nil* and $\widetilde{SL_2\mathbb{R}}$. In the case of trivial stabilizer X is actually a Lie group and from the classification of 3-dimensional Lie algebras it follows that the only geometry of this kind is *Sol*. $\square$

Remark 4.4.15. Since $\mathbb{S}^3$ is the only closed model geometry, it follows from the Geometrization Theorem that a closed simply connected 3-manifold is homeomorphic to $\mathbb{S}^3$, which is the statement of Poincaré Conjecture.

4.4.3 A comparison between JSJ and Geometric Decomposition

The interest of Thurston in the Geometrization program stems from the need of a better understanding of the generic (i.e. atoroidal) pieces arising from a JSJ Decomposition, the other pieces (i.e. Seifert) being well understood and classified as said before. What Perelman actually proved is the following:

Theorem 4.4.16 (Perelman)**.** *Let M be an irreducible atoroidal manifold. Then*

- *(Elliptization Conjecture) if $\pi_1(M)$ is finite then M is elliptic;*
- *(Hyperbolization Conjecture) if $\pi_1(M)$ is infinite then M is hyperbolic.*

It can be proved that an elliptic geometry gives rise to a Seifert structure (see Theorem 2.8 in [Bon02]). As a consequence we see that "being Seifert" and "being atoroidal" are not mutually exclusive properties. On the other hand, from a detailed analysis of the classification of Seifert manifolds, one can deduce that hyperbolic manifolds are not Seifert. This is what actually we can obtain:

Theorem 4.4.17. *The class of Seifert manifolds coincides with the class of manifolds which admit a geometry modelled on one of these: $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL_2\mathbb{R}}$ and Nil. Moreover the geometry type is uniquely determined by the topology of the manifold.*

We can therefore state this form of Torus Decomposition Theorem.

Theorem 4.4.18. *Every irreducible 3-manifold M admits a finite collection of disjoint incompressible tori $T_1, \dots, T_n$ such that the connected components of $M \setminus \setminus (T_1 \cup \cdots \cup T_n)$ are either hyperbolic or Seifert. A minimal such collection is unique up to isotopy.*

The advantage of this formulation is that the resulting pieces are divided into two disjoint classes. The study of 3-manifolds is thus reduced to that of their hyperbolic components.

Remark 4.4.19. Comparing the previous theorem with the list of Seifert geometries, one can see that the *Sol* geometry is missing. This is because the collection of tori in the JSJ decomposition is not in general the same that the one in the Geometric Decomposition; indeed the two procedures of cutting have different objectives: the first aims at pieces which are homotopically simple (i.e. without incompressible tori), whereas the second looks for pieces with geometric structure, so it may happen the they end at different stages. Here below we give an example of this phenomenon.

Example 4.4.20. Let φ an Anosov diffeomorphism of the torus T; by Nielsen-Thurston Classification this amounts to say that φ is not periodic and does not have a fixed curve. Let M be the 3-manifold obtained as the mapping torus of φ. It can be proved that M is a solvmanifold of finite volume, so the Geometric Decomposition of M has just one piece, which is M itself. By the previous discussion M is not Seifert, and of course it is not hyperbolic.

The fiber $T \subset M$ is an incompressible torus so the JSJ Decomposition would find it and cut M open along it. The resulting manifold is the product $T \times I$: it is atoroidal and its interior has a euclidean geometry, but not of finite volume.

4.4.4 Hyperbolic Geometry

As remarked before, Theorem 4.4.18 reduces the study of 3-manifolds to the hyperbolic case. Here we collect some deep facts about hyperbolic 3-dimensional geometry which will be fundamental in the following chapters.

First of all we remark that there are three equivalent ways to think of hyperbolic 3-manifolds and to define them:

- manifolds with a $(Isom(\mathbb{H}^n), \mathbb{H}^n)$-structure,
- Riemannian manifolds of constant sectional curvature -1,
- manifolds obtained as quotients of $\mathbb{H}^n$ by some discrete torsion-free subgroup of $Isom(\mathbb{H}^n)$.

Moreover in dimension 3 there is a canonical identification $Isom(\mathbb{H}^3) \cong SO(3,1) \cong PSL_2\mathbb{C}$. For a discussion of these facts see Chapter B in [BP92].

The following result is maybe the most fundamental one in the study of hyperbolic manifolds; for a detailed treatment we refer again to [BP92], Chapter C.

Theorem 4.4.21 (Mostow's Rigidity Theorem)**.** *Let M, N be hyperbolic manifold of dimension $n \geq 3$. Then every group homomorphism $\varphi : \pi_1(M) \to \pi_1(N)$ is induced by an isometry $f : M \to N$.*

Remark 4.4.22. Notice that the situation is drastically different from that of surfaces: whereas a closed surface of genus $g \geq 2$ admits a continuum of inequivalent hyperbolic metrics, in dimension (at least) 3 the topology uniquely determines the geometric structure. This implies that homotopy equivalent manifolds are actually isometric; in particular we can reduce the study of hyperbolic 3-manifolds to the study of their fundamental groups with no loss of information from a topological point of view. This is quite remarkable, since it paves the way for a massive use of group-theoretic tools in the study of these manifolds. What is even better is that the groups arising as fundamental groups of hyperbolic manifolds enjoy some very nice properties.

Theorem 4.4.23. *If M is a closed hyperbolic manifold, then $\pi_1(M)$ is a hyperbolic group.*

Proof. By the above remarks, M can be realized as ${}^{\mathbb{H}^n}\!/_{\pi_1(M)}$; in particular $\pi_1(M)$ acts properly discontinuously on $\mathbb{H}^n$ by isometries. Since M is compact the action is cocompact. Moreover $\mathbb{H}^n$ is of course a CAT(0) geodesic space. Therefore the result follows from Theorem 2.3.4 and the fact that $\mathbb{H}^n$ contains no isometrically embedded copy of the euclidean plane $\mathbb{E}^2$. □

Remark 4.4.24. Notice that the above property fails in the case of manifolds with boundary. For example consider a knot complement in S^3; such a manifold is often hyperbolic, but the torus boundary gives rise to a $\mathbb{Z} \times \mathbb{Z}$ subgroup of $\pi_1(M)$, which in turn yields a copy of $\mathbb{E}^2$ in the Cayley complex of $\pi_1(M)$.

4.5 Haken Manifolds

In this section we get back to the point of view of cutting along embedded incompressible surfaces and give a general description of the situation for genus $g > 1$ surfaces.

Definition 4.5.1. A Haken manifold is a 3-manifold which is irreducible and contains an incompressible surface.

Remark 4.5.2. The study of these manifolds has a lot to do with the geometric aspects discussed above, since for a long time they have been the starting point for any classic (i.e. pre-Perelman) approach to the Geometrization Conjecture. The reason behind this is that Thurston himself was able to prove a form of hyperbolization for irreducible atoroidal Haken manifolds M with $\chi(\partial M) = 0$ at the beginning of the 1980's, well before Perelman's proof through parabolic equations. Therefore topologists hoped that some kind of reduction to the Haken case could lead to a proof of Geometrization.

An algebraic criterion to check whether a manifold is Haken is the following result; here and in the following H_* denotes singular homology with integer coefficients.

Lemma 4.5.3. *If $H_1(M)$ is infinite then M contains an incompressible surface.*

Proof. From the classification of finitely generated abelian groups we know $H_1(M) \cong \mathbb{Z}^n \oplus T$ for some integer n and some torsion group T; by the hypothesis we have $n \geq 1$, therefore we have an epimorphism $H_1(M) \twoheadrightarrow \mathbb{Z}$, which we may precompose with the Hurewicz map to get an epimorphism $\varphi : \pi_1(M) \twoheadrightarrow \mathbb{Z}$. Since S^1 may be taken as a $K(\mathbb{Z}, 1)$, from 2.1.16 we get a map (unique up to basepoint-preserving homotopies) $f : M \to S^1$ which induces φ on fundamental groups; in particular since φ is non trivial f must be surjective. For any point $p \in S^1$ Lemma 6.5 in [Hem76] implies that f may be homotoped so that each component of $f^{-1}(p)$ is an incompressible surface. □

With some basic algebraic topology one can convert the previous result into an easier and more geometric criterion.

Lemma 4.5.4. *If $\partial M \neq \varnothing$ does not contain only spheres, then $H_1(M)$ is infinite.*

Proof. First of all let $\widehat{M}$ the 3-manifold obtained from M capping off all the boundary spheres with 3-balls. By Seifert-van Kampen and the simply-connectedness of S^2 we compute $\pi_1(\widehat{M}) = \pi_1(M)$ and thus $H_1(\widehat{M}) = H_1(M)$. Therefore we may reduce to the case ∂M contains no spheres.

Since $\partial M \neq \varnothing$ we can take two copies of M and glue them along their boundary, getting a closed manifold M', which is called the double of M. Choosing some triangulation it is easy to see that $\chi(M') = \chi(M) + \chi(M) - \chi(\partial M)$ since the two copies of M inside M' intersect in ∂M. Now, since M' is closed and odd-dimensional, we have by Poincaré Duality and Universal Coefficients Theorem[5] that $\chi(M') = 0$. So we get $\chi(M) = \dfrac{\chi(\partial M)}{2} \leq 0$, since we have reduced to the case ∂M contains no spheres.

Now, if b_i are the Betti numbers of M we have that $b_0 = 1$ since M is connected and $b_3 = 0$ since $\partial M \neq \varnothing$; from the previous discussion we get $1 - b_1 + b_2 \leq 0$ and so $b_1 \geq 1 + b_2 \geq 1$, which implies that $H_1(M)$ is an infinite group. $\square$

Example 4.5.5. Combining the previous lemmas, we can conclude that any (compact, connected and orientable) manifold with sufficiently complicated boundary contains an incompressible surface. For example knot (or link) complements in S^3 contain an incompressible surface, since they have at least one torus boundary component. Another example of Haken manifold is given by surface bundles: see Example 4.2.6.

In complete analogy to the case of essential spheres and incompressible tori, if we have a Haken manifold we can try to cut it open along some incompressible surface. Let M be a 3-manifold and $S \subset M$ an incompressible surface and let M' be the (possibly disconnected) manifold $M \setminus\setminus S$. We say M' is obtained from M by decomposition along S and also write $M \xrightarrow{S} M'$. By construction M' can be naturally identified with a closed submanifold of M. To check that everything goes through as expected we need one more lemma.

[5] See [Hat02], Prop. 2.45 for details.

Lemma 4.5.6. *In the above notation M' is reducible if and only if M is.*

Proof. Suppose M' is reducible; then let T be an essential sphere in $M' \subset M$ that does not bound a ball in M'. If T bounds a ball in M (but not in M') then S must lie inside this ball. But by 4.2.4 any surface in a ball is incompressible, but S is not, so T does not bound a ball in M, which means M is reducible as well.

On the other hand let M be reducible. Choose an essential sphere $T \subset M$ which does not bound a ball in M and that minimizes the number k of components of $S \cap T$. If $k = 0$ then T actually lies in M', which is thus reducible. Otherwise consider an outermost disk D cut by S on T. By incompressibility of S we can find another disk $E \subset S$ such that $\partial E = \partial D$; if $D \cup E$ bounds a ball in M' we can isotope D through E; this operation reduces k, which is absurd by minimality assumption. Otherwise $E \cup D$ is an essential sphere in M' which does not bound a ball in M' and M' is reducible. $\square$

Remark 4.5.7. From 4.2.6, 4.5.6 and the fact that product manifolds $S \times I$ (with S a closed surface) are irreducible, we get another way to see that surface bundles over the circle are Haken manifolds.

Corollary 4.5.8. *Let M be a Haken manifold. Then any (component of the) manifold obtained by decomposition along an incompressible surface is again Haken.*

Proof. Let $S \subset M$ be an incompressible surface and $M \xrightarrow{S} M'$. By 4.5.6 M' is still irreducible. Moreover M' has at least one boundary component which is not a sphere, i.e. that arising from S (which is not a sphere by the given definition of incompressible surface). Then it follows from 4.5.4 and 4.5.3 that M' contains an incompressible surface and is therefore a Haken manifold. $\square$

As a consequence of the previous result, we can keep on splitting the result of the first decomposition and get a (a priori infinite) chain of Haken manifold which get simpler and simpler.

Definition 4.5.9. A series of decomposition $M = M_0 \xrightarrow{S_1} M_1 \xrightarrow{S_2} \dots$ of a Haken manifold along surfaces which are incompressible at each step is called a hierarchy for M provided it terminates after a finite number of steps.

Remark 4.5.10. The nice thing about Haken manifolds is that it is possible to prove that a Haken manifold always has a hierarchy. In the case $\partial M \neq \varnothing$ the above results have to be sharpened a bit to guarantee the surfaces we split along behave nicely with respect to the boundary. By 4.5.4 and 4.5.3 the final step of any hierarchy for M will always be a 3-manifold with only spherical boundary component. This implies that a hierarchy for a Haken manifold can be thought as a procedure to reduce the manifold to a collection of closed 3-balls.

Remark 4.5.11. Haken manifold owe to the previous remark a lot of nice features, which we mention just for the sake of completeness; for example:

- we have an algorithm to check whether a given manifold is Haken;
- we have an algorithm to tell if two Haken manifolds are homeomorphic;
- closed Haken manifold are homotopically rigid: they are determined up to homeomorphism by their fundamental group.

The bad thing about this is that there are non Haken manifolds. So, differently from the previous approaches, not every 3-manifold can be analysed via this kind of decomposition. At least a priori.

Chapter 5

Virtual Fibration of Hyperbolic 3-manifolds

In the last few years the joint work of many mathematicians has shown that the conclusion of the previous chapter is more pessimistic than it should be: 3-manifolds are actually much nicer than expected. In 1968 Waldhausen implicitly proposed the following conjecture.

Conjecture 5.1 (Virtually Haken Conjecture)**.** *A compact, orientable, irreducible* 3*-manifold with infinite fundamental group is virtually Haken, i.e. it has a finite covering space that is a Haken manifold.*

As we have seen in 4.4.3, Geometrization implies that an irreducible atoroidal manifold has infinite fundamental group if and only if it is hyperbolic, therefore we may restrict to the hyperbolic case. Moreover, 4.5.4 let us reduce to the case of closed manifolds.

Conjecture 5.2 (Virtually Haken Conjecture - Thurston's Hyperbolic statement)**.** *A closed hyperbolic* 3*-manifold is virtually Haken.*

This is exactly what Thurston posed as question 16 in his notorious article [Thu82]. In the same article he posed another problem; before stating it we introduce the main definition of this chapter.

Definition 5.0.12. A 3-manifold M is said to fiber over the circle if it is a bundle over S^1 with fiber some fixed surface. In the following we also just say that M is a fibered[1] 3-manifold.

This is the statement of question 18 in [Thu82].

Conjecture 5.3 (Virtually Fibered Conjecture). *A closed hyperbolic 3-manifold is virtually fibered.*

We recall from 4.2.6 that being a surface bundle over the circle is equivalent to being a mapping torus for some homeomorphism of the fiber surface and that in such a bundle the fiber is an incompressible surface, which means that

$$\text{Virtually Fibered Conjecture} \Longrightarrow \text{Virtually Haken Conjecture}$$

As mentioned above, a solution to the Virtually Haken Conjecture (and a fortiori the Virtually Fibered Conjecture) would lead to a proof of the Geometrization Conjecture without any appeal to differential equation (i.e. Perelman's work); anyway, as Thurston himself admitted in [Thu82]

> Unfortunately, there seems to be little prospect of finding such finite-sheeted coverings without first knowing the manifold is hyperbolic.

So far Thurston has been right: proofs of both conjectures have recently been found for the hyperbolic case ([Ago12]) assuming Geometrization and heavily relying on the properties of the hyperbolic geometry of such manifolds and their fundamental groups. In this chapter we describe how the techninques of the previous chapters have been employed to construct a fibered structure on a suitable covering space of a closed hyperbolic 3-manifold. In doing this we also introduce some complementary notions of 3-dimensional topology. Here is a list of the main steps of the proof of the Virtually Fibered Conjecture: let M be a closed hyperbolic 3-manifold

1. find "enough" immersed π_1-injective surfaces in M;

[1] Notice that by a count of the dimensions involved, a 3-dimensional bundle can either be a bundle with base a surface and fiber a circle, which is the case of Seifert manifolds as seen in 4.3.1, or a bundle with base a circle and fiber a surface. In the following "fibered" will always refer to this second meaning.

2. the subgroups they induce in $\pi_1(M)$ allow to construct a cube complex X with $\pi_1(X) = \pi_1(M)$;

3. since $\pi_1(M)$ is a hyperbolic group, this cube complex turns out to be compact and (virtually) special;

4. as a result, $\pi_1(M)$ virtually embeds in some RAAG;

5. RAAGs satisfy a certain technical algebraic condition (the RFRS condition);

6. a 3-manifold with RFRS fundamental group admits a fibered cover.

5.1 Cubulation of Groups and Manifolds

By "cubulation of groups" we mean a combinatorial construction which was first introduced by Sageev in [Sag95] and then in [Sag97]; this construction takes a group G with a suitable family of subgroups and produces a cube complex with a G-action. Bergeron and Wise in [BW10] then exploit this construction to "cubulate" a closed hyperbolic 3-manifold (i.e. to cubulate its fundamental group), heavily relying on the work [KM09] of Kahn and Markovic on the existence of almost geodesic surfaces in such a manifold.

5.1.1 Codimension-1 Subgroups

In this paragraph we describe the fundamental ideas behind the cubulation of groups.

Definition 5.1.1. Let G be a finitely generated group and H a subgroup. In 2.3.6 we have defined the space of ends of G as the space of ends of its Cayley graph with respect to some generating set. We now define the ends of G relative to H, denoted $Ends(G, H)$, to be the space of ends of the metric space obtained as the quotient of the Cayley graph of G modulo the action of H. We denote by $e(G, H)$ the cardinality of $Ends(G, H)$.

We remark that this definition is well posed since the space of ends is a quasi-isometric invariant of proper geodesic spaces.

Definition 5.1.2. A subgroup H of a finitely generated group G is a codimension-1 subgroup if $e(G, H) > 1$.

Remark 5.1.3. The idea behind this definition is that the subgraph of the Cayley graph of G corresponding to H is large enough to disconnect the graph at infinity. Basic examples are given by $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{n+1}$ for every n. Less trivial examples are given by the subgroups corresponding to an embedded essential loop on a surface or to an incompressible surface in a 3-manifold.

Sageev in [Sag95] shows how to go from a codimension-1 subgroup H to a cube complex X with a G-action, which should be considered as dual to H. Fix some generating set S for G and call Γ the associated Cayley graph and $\Gamma_H := {}^{\Gamma}\!/_{H}$ its quotient by the action of H. Since H is a codimension-1 subgroup, we can find a compact $K \subset \Gamma_H$ such that $\Gamma_H \setminus K$ is disconnected with two unbounded components. Let Z be such a component of $\Gamma_H \setminus K$, Y its preimage in Γ and Y^c the complement of Y in Γ. Finally we define a collection of subsets of Γ by

$$\Sigma = \{gY|g \in G\} \cup \{gY^c|g \in G\}$$

This is a partially ordered set with respect to inclusion and comes equipped with a natural G-action by left multiplication. We are now ready to define a cube complex C associated to the subgroup H. We begin with low dimensional cells. Vertices can be roughly thought as utrafilter in this poset.

Definition 5.1.4. A vertex of C is given by a subset $V \subseteq \Sigma$ such that

- $\forall\ A \in \Sigma$ exactly one of A and A^c belongs to V,
- if $A, B \in \Sigma$, $A \in V$ and $A \subset B$ then $B \in V$.

Example 5.1.5. For every $g \in G$ let $V_g = \{A \in \Sigma \mid g \in A\}$. Then V_g is a vertex. We call a vertex obtained in this way a principal vertex.

We now turn to edges.

Definition 5.1.6. Two vertices V and W are joined by an edge in C if and only if $\exists\ A \in V$ such that $(V \setminus A) \cup A^c = W$.

Remark 5.1.7. Sageev gives some equivalent formulations for this. For example, we have that $(V \setminus A) \cup A^c = W$ if and only if $|V \setminus W| = |W \setminus V| = 1$; moreover $(V \setminus A) \cup A^c$ is a vertex if and only if A is minimal in V with respect to the induced order.

This defines a graph C.

Definition 5.1.8. We now define X^1 to be the subgraph given by the union of the components of C which contain some principal vertex V_g.

This restriction is needed to forget some undesired vertices. Anyway the selected subgraph turns out to be still connected.

Example 5.1.9. Let $G = \mathbb{Z}$, $H = \{0\}$ and $\Gamma =$Cay$(\mathbb{Z}, \{1\})$. Since the action of H on Γ is trivial we have $\Gamma_H = \Gamma$, so we can disconnect it in two unbounded components just removing any point. For every $n \in \mathbb{Z}$ the principal vertex V_n is given by the collection of intervals which contain n, i.e intervals of the form $\{k \leq m\}$ for some $m \geq n$ or of the form $\{k \geq m\}$ for some $m \leq n$. Notice that the induced order on V_n has two minimal elements, namely $]-\infty, n]$ and $[n, +\infty[$; by 5.1.7 this means that V_n is joined by an edge to V_{n-1} and V_{n+1}. Therefore the graph C has a subgraph isomorphic to Cay$(\mathbb{Z}, \{1\})$. However the collection of sets of the form $\{k \geq m\}$ for some m gives a vertex V_+ which has no minimal element and is thus disconnected from the above chain. This is a sort of vertex at $+\infty$ and of course one has an analogous V_- at $-\infty$; these two vertices correspond to points in $\partial\mathbb{Z}$. In this example C contains V_+ and V_- but X^1 does not. Also notice that the action of $\mathbb{Z}$ on C by translations is free on principal vertices but has V_+ and V_- as fixed points.

After restricting to X^1, we then recursively complete the construction by glueing in an n-cube every time we see the boundary of an n-cube, and call X the resulting cube complex. It is clear from the construction that X is an NPC cube complex; Sageev also proved the following (see Theorem 3.7 in [Sag95]).

Theorem 5.1.10. *X is a simply connected cube complex.*

Thus X is a CAT(0) cube complex.

Remark 5.1.11. Following Sageev, we have carried out the construction just for one subgroup. It is possible to replicate the above for any finite collection $\{H_1, \ldots, H_n\}$ of codimension-1 subgroups of G: just take $\Sigma = \cup_{i=1}^{n} \Sigma_i$, where Σ_i is the individual poset associated to each H_i as before. The construction goes through with no substantial modifications.

As pointed out before, G naturally acts on Σ by left multipliation, and of course this gives an action on the vertices of X in the obvious way; then we may combinatorially extend the action to the whole cube complex X. However, in full generality, we do not have a nice control of this action. Anyway if we put some restriction on the group and the collection of subgroups then something can be said. We describe now some results about the class of examples which we are interested in for the application to 3-manifolds.

Definition 5.1.12. A subspace X of a geodesic metric space Y is quasiconvex if any geodesic in Y with endpoints in X remains at bounded distance from X.

Definition 5.1.13. Let G be a finitely generated group with generating set S. A subgroup H is quasiconvex with respect to S if it is a quasiconvex subset of $Cay(G, S)$.

We have a nice characterization of quasiconvexity in hyperbolic groups (see Corollary III.Γ.3.6 in [BrH99]).

Lemma 5.1.14. *Let G be a hyperbolic group. Then a subgroup H is quasiconvex (with respect to some generating set) if and only if the inclusion $H \hookrightarrow G$ is a quasi-isometric-embedding.*

Remark 5.1.15. In particular it follows that

- the notion of quasiconvexity does not depend on the choice of the generating set;
- a quasiconvex subgroup of a hyperbolic group is itself hyperbolic.

In [Sag97] Sageev proves the following.

Theorem 5.1.16. *If G is hyperbolic and H a quasiconvex codimension-*1 *subgroup, then the action of G on the associated cube complex X is cocompact.*

If we could say that the action was also proper and free, then G would act by a covering action, which would imply that $G = \pi_1\left({}^{X}\!/_{G} \right)$, i.e. G would be realized as the fundamental group of a compact cube complex. Even if the action is not free, the quotient ${}^{X}\!/_{G}$ can still be considered as an orbihedron (a combinatorial analogue of an orbifold, in the sense of [Hae91]). In general we say that G is cubulated when it is equipped with a faithful proper cocompact action on a CAT(0) cube complex.

Bergeron and Wise in [BW10] proposed a criterion to check when the action is also proper. There they prove the following.

Theorem 5.1.17. *Let G be a hyperbolic group. Suppose that for each couple of points at infinity $p \neq q \in \partial G$ there exists some quasiconvex codimension-1 subgroup H such that ∂H separates p and q in ∂G. Then we can find a finite collection $\{H_1, \dots, H_n\}$ of quasiconvex codimension-1 subgroups such that the action of G on the associated cube complex is proper and cocompact.*

Although it may look quite difficult to check the hypothesis of the above theorem, it turns out that this condition is (non trivially!) satisfied in the case G is the fundamental group of a closed hyperbolic 3-manifold. This follows from a deep result of Kahn and Markovic which we describe in the next section.

5.1.2 Surface Subgroups

Bergeron and Wise have shown how to use their boundary criterion in the context of hyperbolic 3-manifolds. So far, the usefulness of cube complexes in the study of such manifolds heavily relies on the work of Kahn and Markovic on the existence of almost geodesic surfaces in a closed hyperbolic 3-manifold, see [KM09].

Informally speaking, their main result is that given a closed hyperbolic 3-manifold, we can always find a *dense* collection of immersed π_1-injective surfaces inside it. Here *dense* should be interpreted according to the following version of the result, which is roughly the one given in [BW10].

Theorem 5.1.18 (Kahn, Markovic)**.** *Let M be a closed hyperbolic 3-manifold and let $\widetilde{M} \cong \mathbb{H}^3$ be its universal cover. Let C be a great circle in $\partial\mathbb{H}^3$. Then* *we can find a sequence of immersed π_1-injective surfaces $S_n \to M$ such that $\partial\widetilde{S}_n$ converges to C pointwise in $\partial\mathbb{H}^3$, where $\widetilde{S}_n \cong \mathbb{H}^2 \subset \mathbb{H}^3$ denotes the universal cover of S_n.*

Remark 5.1.19. Before describing why this is relevant in the cubulation of groups and manifolds, we observe that the result of Kahn and Markovic gives a solution to another longstanding conjecture of Waldhausen, the Surface Subgroup Conjecture. This conjecture asked the following: if a closed, irreducible 3-manifold M has infinite fundamental group, then is it true that $\pi_1(M)$ contains a subgroup isomorphic to the fundamental group of a closed surface? After Geometrization, the only case left open was that of hyperbolic manifolds, thus the above result affirmatively solves this problem.

From the above result Bergeron and Wise then deduce the following.

Corollary 5.1.20. *In the same notations as above, let $p, q \in \partial\widetilde{M}$ be a pair of distinct points. Then we can find an immersed π_1-injective surface S such that $\partial\widetilde{S}$ separates p and q.*

Proof. Just take any geodesic joining p to q and let C be a great circle orthogonal to this geodesic through its midpoint. Then the previous theorem gives a sequence of surfaces S_n such that in particular $\partial\widetilde{S}_n$ converges to C pointwise. Since C itself separates the two points by construction, we can conclude that at least for n big enough S_n separates them too. $\square$

As a combination of 5.1.17 and 5.1.20 we get the following theorem.

Theorem 5.1.21. *Let M be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ acts freely properly and cocompactly on a CAT(0) cube complex. Equivalently, $\pi_1(M)$ is the fundamental group of a compact NPC cube complex.*

The freeness of the action in this case can be deduced from the fact that hyperbolic groups arising as fundamental groups of closed hyperbolic 3-manifolds are torsion-free; this in turn is a consequence of the fact that these manifolds are finite-dimensional Eilenberg-MacLane spaces for their fundamental group (compare Proposition 2.45 in [Hat02]).

Of course one has to check that the subgroups coming from the inclusions of the surfaces constructed by Kahn and Markovic are quasiconvex and of codimension 1; this follows from the fact that these surfaces are not just π_1-injective, but enjoy some additional geometric properties: they can be chosen to be arbitrarily small perturbations of totally geodesic surfaces.

When the conclusion of the theorem holds, we say that the mentioned cube complex is a cubulation of M, or also that M has been cubulated.

Remark 5.1.22. Notice that the cubulation we have described allows us to see the fundamental group of a (closed hyperbolic) manifold of dimension 3 as the fundamental group of a cube complex that in general will have much more than 3 dimensions. This is a warning: the term "cubulation" in this context has definitely not the same meaning as "triangulation", i.e. it does not refer to a combinatorial complex with a geometric realization which is homeomorphic to the manifold we started with. This may at first sound not very smart, but it turns out to be a great deal to get a very organized structure like a cube complex instead of a wild 3-manifold, even if we have to pay a lot of extra dimensions.

Anyway we remark that the universal cover of a closed hyperbolic n-manifold is $\mathbb{H}^n$ which is contractible and that also the universal cover of an NPC cube complex is a CAT(0) space (by 3.4.4), and then contractible by 1.2.7; this implies that both hyperbolic manifolds and NPC cube complexes are Eilenberg-MacLane spaces for their fundamental groups. An application of 2.1.17 and 5.1.21 shows then that a closed hyperbolic 3-manifold and its cubulation are at least homotopy equivalent.

5.1.3 Agol's Virtual Compact Special Theorem

We have studied cube complexes in Chapter 3 and there is no doubt that the most amazing result about them is that the fundamental group of a compact special cube complex virtually embeds in a RAAG, i.e. has a finite index subgroup which embeds in a RAAG. If we want to exploit this result, then we have to ensure that the cubulation of a closed hyperbolic 3-manifold described above is special, at least virtually.

From a chronological point of view, this has been the last issue to be addressed in the proof of the Virtually Haken and Fibered Conjectures. In [Wi11] Wises poses the problem in the following form (Conjecture 19.5):

Conjecture. *Let G be a hyperbolic group acting properly and cocompactly on a CAT(0) cube complex X. Then G has a finite index subgroup acting on X with special quotient.*

This is of course equivalent to saying that G is the fundamental group of a compact NPC cube complex which has a finite cover which is special. This condition is also expressed by saying that G is virtually special. This is why Agol's theorem is often referred to as the "Virtual Compact Special Theorem". Agol proved this conjecture only a few months ago in [Ago12].

One of the major observations (due to Wise, [Wi11]) in this context is that virtually special hyperbolic groups can be recursively axiomatized as a certain class of group closed under some well understood operations. First of all we recall two classic constructions in combinatorial group theory.

Definition 5.1.23. Let F, G and H be groups and let $g : F \to G$ and $h : F \to H$ be two homomorphisms. The quotient group of the free product $G*H$ obtained imposing $g(x) = h(x)$ for every $x \in F$ is called the amalgamated free product of G and H over F and is denoted by $G \underset{F}{*} H$.

Definition 5.1.24. Let $G = \langle S|R\rangle$ be a group, H another group and choose two embeddings $f : H \hookrightarrow G$ and $g : H \hookrightarrow G$ of H in G. Take any symbol $t \notin S$. The group with presentation $\langle S, t|R, tf(h) = g(h)t \ \forall\ h \in H\rangle$ is called the HNN extension of G over H relative to fg^{-1}. When we do not need to specify the map, we just denote it by $G\underset{H}{*}$ and say it is an HNN extension of G over H.

Remark 5.1.25. The above algebraic constructions are motivated by concrete topological operations: the amalgamated free product arises in the statement of Seifert-van Kampen Theorem, whereas the HNN extensions correspond to the fundamental group of a mapping torus. This analogy is actually quite strong and one could indeed develop the theory either in an abstract (=combinatorial) fashion or in a more concrete way, realizing the groups involved as fundamental groups of some Eilenberg-MacLane spaces and then performing the mentioned topological constructions on the resulting spaces.

The following definition is due to Wise ([Wi11], Definition 11.5).

Definition 5.1.26. We denote by $\mathcal{QVH}$ and call Quasi-Convex Virtual Hierarchy Class the smallest class of groups which consists of hyperbolic groups, contains the trivial group and such that:

1. if $A, C \in \mathcal{QVH}$, B is finitely generated and is quasi-isometrically embedded in $A \underset{C}{*} B$, then $A \underset{C}{*} B \in \mathcal{QVH}$
2. if $A \in \mathcal{QVH}$, B is finitely generated and is quasi-isometrically embedded in $A \underset{B}{*}$, then $A \underset{B}{*} \in \mathcal{QVH}$
3. if $H \in \mathcal{QVH}$ is a finite index subgroup of G, then $G \in \mathcal{QVH}$

A motivating example of a group in $\mathcal{QVH}$ is given by the fundamental group of a hyperbolic 3-manifold which contains an embedded π_1-injective surface. The idea of the above definition is that a group is in $\mathcal{QVH}$ if it can be obtained by a sequence of elementary moves starting from the trivial group and proceeding in a geometrically controlled way. This is reminiscent of the notion of hierarchy for a Haken manifold, and $\mathcal{QVH}$ should indeed be thought as an analogous in geometric group theory of hyperbolic Haken manifolds.

The interest in this class of groups lies in the following result which Agol proves as Theorem A.42 in the appendix of [Ago12] extending Wise's Theorem 13.3 of [Wi11].

Theorem 5.1.27. *Let G be a hyperbolic group. Then $G \in \mathcal{QVH}$ if and only if G is virtually special.*

Therefore Agol's Virtual Compact Special Theorem is proved by showing that a hyperbolic group G acting properly and cocompactly on a CAT(0) cube complex X is in $\mathcal{QVH}$. More precisely this is achieved through an inspection of the combinatorics of hyperplanes in ${}^{X}\!/_{G}$.

Combining the Virtual Compact Special Theorem 5.1.3, the theorem on cubulation of manifolds 5.1.21 and the fact that the fundamental group of a closed hyperbolic 3-manifold is hyperbolic (4.4.23), we get the following result.

Theorem 5.1.28. *Let M be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ is virtually the fundamental group of a compact special cube complex.*

We have already seen how to go from compact special cube complexes to RAAGs. Thus we know that under the hypothesis of the theorem $\pi_1(M)$ is virtually embedded in a RAAG. In the next section we will describe how to go from there to fibrations.

5.2 From RAAG to Fibrations

In 3.3.10 we have seen that the fundamental group of a compact special cube complex virtually embeds in a RAAG; by the previous section about cubulation, especially Theorem 5.1.28, we can conclude that also the fundamental group of a closed hyperbolic 3-manifold virtually embeds in a RAAG. These groups enjoy a lot of nice algebraic properties and in this section we focus on one that allows to find fibered covers.

5.2.1 Residually Finite Rationally Solvable Groups

The main algebraic property we are interested in was introduced by Agol in [Ago08].

Definition 5.2.1. Let G be a group. We inductively define its rational derived series as follows

- set $G^{(1)} := [G, G]$ and $G_r^{(1)} := \{x \in G | \exists\ k \neq 0 : x^k \in G^{(1)}\}$,
- then set $G_r^{(n+1)} := (G_r^{(n)})_r^{(1)}$.

This series should be thought as being obtained from the classic derived series[2] of G by killing torsion at each step. The standard way to kill torsion is to take a tensor product with $\mathbb{Q}$ (over $\mathbb{Z}$), and this motivates the name of the series, as the following lemma explains, in terms of the homology[3] of G.

Lemma 5.2.2. $G_r^{(1)} = \ker \left\{ f : G \to {}^{G}\!/_{G^{(1)}} \otimes \mathbb{Q} \cong H_1(G, \mathbb{Q}) \right\}$

[2]This is defined by induction starting from $G^{(1)} := [G, G]$ and then setting $G^{(n+1)} := [G^{(n)}, G^{(n)}]$. In both cases one could also define $G^{(0)} := G$ for convenience.

[3]The definition of group homology was given in 2.1.18 for integral homology, but of course it readily extends to homology with arbitrary coefficients.

Proof. First of all we describe the natural map f in the statement. This is just defined as $f(x) := [x] \otimes 1$. Then we verify the isomorphism in the codomain of the map f. Let X be any $K(G, 1)$; by the Universal Coefficients Theorem we get

$$H_1(G, \mathbb{Q}) = H_1(X, \mathbb{Q}) \cong H_1(X) \otimes \mathbb{Q} \cong (\pi_1(X))_{ab} \otimes \mathbb{Q} \cong G_{ab} \otimes \mathbb{Q} = {}^{G}\!\diagup_{G^{(1)}} \otimes \mathbb{Q}$$

Then we check that for every integer $k \neq 0$ we have

$$[x] \otimes 1 = [x] \otimes \left(k.\frac{1}{k}\right) = (k.[x]) \otimes \frac{1}{k} = [x^k] \otimes \frac{1}{k}$$

which implies that $f(x) = 0$ if and only if $\exists\ k \neq 0 : x^k \in G^{(1)}$, which by definition means that $x \in G_r^{(1)}$. $\square$

Killing torsion does not affect the usual properties of the derived series. We check these in the following lemmas.

Lemma 5.2.3. *The rational derived series of a group is a descending normal series, i.e. we have $G \trianglerighteq \dots G_r^{(n)} \trianglerighteq G_r^{(n+1)} \trianglerighteq \dots$; moreover, ${}^{G_r^{(n)}}\!\diagup_{G_r^{(n+1)}}$ is an abelian torsion-free group.*

Proof. This readily follows from the definition and Lemma 5.2.2

$$G_r^{(n+1)} = (G_r^{(n)})_r^{(1)} = \ker\{f : G_r^{(n)} \to H_1(G_r^{(n)}, \mathbb{Q})\}$$

Since every group in the series is normal in the preceding one, we can form the quotient group and it is abelian and torsion-free because we quotient out all the elements that up to some power are commutators. $\square$

For a general normal series we cannot guarantee that the n-th step will be a normal subgroup of the whole group. But here we actually have the following fact.

Lemma 5.2.4. *At each step of the rational derived series we have $G_r^{(n)} \trianglelefteq G$.*

Proof. We proceed by induction on n. From Lemma 5.2.2 we know this is true for $n = 1$. Now take $y \in G_r^{(n+1)}$ and $x \in G$, and choose some $k \neq 0$ such that $y^k \in (G_r^{(n)})^{(1)} = [G_r^{(n)}, G_r^{(n)}]$. Then we have that

$$(xyx^{-1})^k = xy^kx^{-1} \in x[G_r^{(n)}, G_r^{(n)}]x^{-1} \subseteq$$

$$\subseteq [xG_r^{(n)}x^{-1}, xG_r^{(n)}x^{-1}] \subseteq [G_r^{(n)}, G_r^{(n)}] = (G_r^{(n)})^{(1)}$$

where the last inclusion holds since by induction $G_r^{(n)} \trianglelefteq G$. But the above chain of inclusions implies that $xyx^{-1} \in G_r^{(n+1)}$. □

Lemma 5.2.5. *$G/G_r^{(n)}$ is a solvable group.*

Proof. Lemma 5.2.4 guarantees we can form the quotient. By Lemma 5.2.3 $G \trianglerighteq \cdots \trianglerighteq G_r^{(n)} \trianglerighteq \dots$ is a normal series for G with abelian intermediate quotients. Therefore its reduction modulo $G_r^{(n)}$ gives a normal series with abelian intermediate quotients and finite length for $G/G_r^{(n)}$; thus it is solvable. □

We are now ready to give the main definition of this section.

Definition 5.2.6. A group G is Residually Finite Rationally Solvable (or RFRS) if it admits a descending sequence of subgroups $G = G_0 \supset G_1 \supset \dots$ that satisfy the following properties

1. $G_n \trianglelefteq G$
2. $\bigcap_{n \geq 0} G_n = 1$
3. $[G : G_n] < \infty$
4. $(G_n)_r^{(1)} \subseteq G_{n+1}$

Such a sequence will also be called a RFRS series for G in the following.

Lemma 5.2.7. *We have that $G_r^{(n)} \subseteq G_n$ and that G/G_n is solvable.*

Proof. We proceed by induction. $G_r^{(1)} = (G_0)_r^{(1)} \subseteq G_1$ by property 4 in the definition. Then $G_r^{(n+1)} = (G_r^{(n)})_r^{(1)} \subseteq (G_n)_r^{(1)}$ by inductive hypothesis and this lies in G_{n+1} again by property 4. This proves the first part of the statement. By Lemma 5.2.5 we know that $G/G_r^{(n)}$ is solvable; therefore the first part of the statement implies the second. □

This essentially means that we can think of the descending series in the definition of a RFRS group as somehow obtained by enlarging a little bit and in a controlled way the rational derived series of the group.

When one wants to check if a group is RFRS, it turns out that he can avoid checking the normality conditions, since one can always extract a normal subseries; we now show how this can be done.

Definition 5.2.8. Given a subgroup H of G, the normal core $Core(H)$ of H in G is defined to be the biggest normal subgroup of G contained in H, i.e. $Core(H) = \bigcap\limits_{g\in G} gHg^{-1}$.

Remark 5.2.9. Given a group G and a subgroup H, G acts on the set of left cosets ${}^{G}\!/\!_{H}$ by left multiplication. This gives a map $G \to Sym\left({}^{G}\!/\!_{H}\right)$, where $Sym(X)$ denotes the permutation group over a set X. The stabilizer of a left coset gH is obviously given by gHg^{-1}, thus we see that $Core(H) = \ker\left(G \to Sym\left({}^{G}\!/\!_{H}\right)\right)$. If H has finite index in G we can conclude that $[G : Core(H)] = \left|Sym\left({}^{G}\!/\!_{H}\right)\right| = [G : H]!$.

Lemma 5.2.10. *If G has a sequence $G = G_0 \supset G_1 \supset \ldots$ of finite index subgroups such that $\bigcap\limits_{n\geq 0} G_n = 1$ and $(G_n)_r^{(1)} \subseteq G_{n+1}$ then G is RFRS.*

Proof. Define $\widehat{G}_n := Core(G_n)$. By definition, this is still a descending series and each of its elements is a normal subgroup of G, i.e. property 1 is satisfied. Then we verify property 2

$$\bigcap_{n\geq 0}\widehat{G}_n = \bigcap_{n\geq 0}\bigcap_{g\in G} gG_ng^{-1} = \bigcap_{g\in G}\bigcap_{n\geq 0} gG_ng^{-1} = \bigcap_{g\in G} g\left(\bigcap_{n\geq 0} G_n\right)g^{-1} = 1$$

By the above remark, we can compute $[G : \widehat{G}_n] = [G : G_n]! < \infty$, which proves property 3.

Finally, we check property 4, i.e. $(\widehat{G}_n)_r^{(1)} \subseteq \widehat{G}_{n+1}$. Indeed we have

$$(\widehat{G}_n)_r^{(1)} = \left(\bigcap_{g\in G} gG_ng^{-1}\right)_r^{(1)} \subseteq \bigcap_{g\in G}(gG_ng^{-1})_r^{(1)} \subseteq \bigcap_{g\in G} gG_{n+1}g^{-1} = \widehat{G}_{n+1}$$

where the last inclusion comes from the fact that $(gG_ng^{-1})_r^{(1)} = g(G_n)_r^{(1)}g^{-1}$ and this lies in $gG_{n+1}g^{-1}$ by hypothesis. □

We conclude this section observing that being RFRS is a property that passes to any subgroup, irrespective of whether they have finite index.

Lemma 5.2.11. *Let G be RFRS and H a subgroup; then H is RFRS.*

Proof. Let $\{G_n\}$ be a descending series of subgroups of G as in Definition 5.2.6 and let $H_n := H \cap G_n$; then $\{H_n\}$ is a descending series of subgroups of H. Properties 1 and 2 are immediately verified. Then we observe that we have a map $H \to G \to {}^{G}\!/\!_{G_n}$ whose kernel is given by $H \cap G_n = H_n$, therefore we get an injection ${}^{H}\!/\!_{H_n} \hookrightarrow {}^{G}\!/\!_{G_n}$ into a group which is finite since G_n has finite index in G by definition; this implies that H_n has finite index in H, i.e. property 3. Finally we have that $H_n \subseteq G_n$ implies that $(H_n)_r^{(1)} \subseteq (G_n)_r^{(1)}$ and this sits in G_{n+1} by hypothesis, therefore $(H_n)_r^{(1)} \subseteq H_{n+1}$ and 4 is proved. □

5.2.2 Right-Angled Coxeter Groups

We now introduce a class of groups which is strictly connected to RAAGs, which where introduced in section 3.3.1.

Definition 5.2.12. Let Γ be a simplicial graph. The right-angled Coxeter group (RACG in the following) associated to Γ is the group with the following presentation:

$$C(\Gamma) = \langle x_i \in \Gamma^0 \mid x_i^2 \ \forall\ x_i \in \Gamma^0\ , [x_i, x_j] \text{ if } \{x_i, x_j\} \in \Gamma^1 \rangle$$

This can be obviously obtained as the quotient of the RAAG $A(\Gamma)$ on the same graph by the normal subgroup generated by the squares of the generators. RACGs (which are an example of the general class of Coxeter groups) have been intensively studied since the beginning of the last century, because they have many connection with reflection groups and the theory of semisimple Lie algebras. Therefore it is quite interesting that RAAGs can be embedded as subgroups of suitable RACGs: one hopes that in this way some of the well-established properties of RACGs can be inferred for RAAGs as well. This is actually the case for the RFRS condition, and this is what we want to describe in this section.

First of all we show how to embed a RAAG in a RACG. The basic construction is given by this example.

Example 5.2.13. Let Γ be the discrete graph on 2 points. Then $C(\Gamma) = \mathbb{Z}_2 * \mathbb{Z}_2 = Dih_\infty$ is the infinite dihedral group. This group can be concretely realized as the subgroup of isometries of $\mathbb{R}$ generated by the reflections in 0 and 1

$$\rho_0(x) = -x \text{ and } \rho_1(x) = 2 - x$$

This is the easiest (non finite[4]) example of a RACG. If Γ' is the singleton graph, then $A(\Gamma') = \mathbb{Z}$ and this can be embedded in $C(\Gamma)$ sending its generator to the composition of the two reflections $\rho_1\rho_0(x) = x + 2$, which is a translation of $\mathbb{R}$. Notice that the subgroup generated by this translation has index 2 in $C(\Gamma)$.

Proposition 5.2.14. *Let Γ be a simplicial graph and $A(\Gamma)$ the associated RAAG. Then there exists a simplicial graph Γ' with associated RACG $C(\Gamma')$ and a monomorphism $A(\Gamma) \hookrightarrow C(\Gamma')$.*

Proof. We construct Γ' as follows. The vertex set is given by $\Gamma \times \{0, 1\}$ and we give it coordinates (v, i) with $v \in \Gamma^0$ and $i \in \{0, 1\}$. The two vertices (v, i) and (w, j) span an edge in Γ' if and only if v and w span an edge in Γ. This should be thought as a doubling of Γ completed with the necessary edges. The map defined in the previous example allows to map (injectively) each generator v of $A(\Gamma)$ into the product of the corresponding generators $(v, 0)$ and $(v, 1)$ of $C(\Gamma')$; the additional edges ensure that this can be extended to a group homomorphism. □

Remark 5.2.15. In the example above the RAAG is embedded as a finite index subgroup and the proof of the proposition is a direct extension of that idea to the case of higher rank groups. However when the RAAG has more than one generator, in general this construction gives a subgroup of infinite index; for example consider the case when Γ is the discrete graph on two points: in this case the cosets of the image subgroup are in correspondence with $\mathbb{Z}_2 * \mathbb{Z}_2$. For the sake of completeness, we mention that Davis and Januszkiewicz have found a more sophisticated construction to achieve a finite index embedding. The underlying philosophy is the same described here, but they observe that the group $C(\Gamma')$ constructed above is bigger than necessary: they consider

[4]Of course $\mathbb{Z}_2$ is a RACG over the singleton graph with only one point; but finite groups cannot contain RAAG as subgroups since these are infinite groups, so we are not interested in finite RACG.

the graph Γ'' obtained from our Γ' by making the subgraph $\Gamma \times \{0\}$ a complete graph (i.e. any two points span an edge) and the associated RACG $C(\Gamma'')$. This is of course a smaller group than $C(\Gamma')$ and the map described above maps $A(\Gamma)$ onto a finite index subgroup. The non trivial point (which they prove in [DJ00]) is that this is still an embedding. However, we will not need this construction in the following.

The reason why we care about this embedding is the following result, which is due to Agol, see Theorem 2.2 in [Ago08] or Lemma 5.18 in [Kob13].

Theorem 5.2.16 (Agol)**.** *Finitely generated RACGs are virtually RFRS.*

Sketch of proof. Let G be a finitely generated RACG. There is a classic representation-theoretic construction that realizes G as a group of reflections in orthogonal hyperplanes in some euclidean space, with quotient a compact polyhedron $\mathcal{C}$ (actually an orbihedron). See [Dav08] for details. The abelianization of G is a direct product of copies of $\mathbb{Z}_2$, thus its kernel induces a finite covering space of $\mathcal{C}$. Moreover, a sequence of 2-sheeted covering spaces of $\mathcal{C}$ can be obtained through reflections in top dimensional faces. Playing with these covers, Agol has been able to identify a finite index RFRS subgroup of G. □

By 5.2.11 we know that subgroups of RFRS groups are RFRS too. To conclude that RAAG are RFRS we just need to check that everything works fine for virtually RFRS groups.

Corollary 5.2.17. *Finitely generated RAAGs are virtually RFRS.*

Proof. Let G be a finitely generated RACG. By 5.2.14 we can find a RACG $\widehat{G}$ such that $G \hookrightarrow \widehat{G}$. By 5.2.16 $\exists\ \widehat{H} \subseteq \widehat{G}$ which is a finite index RFRS subgroup. Then $H := \widehat{H} \cap G$ is a subgroup of G which is RFRS by 5.2.11, since it is also a subgroup of $\widehat{H}$ which is RFRS. If we consider the map $G \hookrightarrow \widehat{G} \to {}^{\widehat{G}}\!/_{\widehat{H}}$, we see that its kernel is given by $G \cap \widehat{H} = H$, which implies that we have an injection ${}^{G}\!/_{H} \hookrightarrow {}^{\widehat{G}}\!/_{\widehat{H}}$, thus H has finite index in G because $\widehat{H}$ has finite index in $\widehat{G}$. □

We already know that the fundamental group of a hyperbolic 3-manifold is virtually embedded in a RAAG. By this theorem it is also virtually RFRS. Then the strategy now is to show that a manifold with RFRS fundamental group can be cut and covered in a suitable way, so that at each step we have some kind of measure of how far the different pieces are from being fibered. In the next paragraph we introduce the basic machinery involved in this approach.

5.2.3 Thurston's Norm on Homology

Now we describe a norm on the homology with real coefficients of a 3-manifold introduced by Thurston in [Thu86]. This is based on a notion of complexity for surfaces and turns out to be a powerful tool in the study of fibrations of 3-manifolds. Beyond Thurston's original paper, the topics discussed here are treated in [Cal07] and [CaC00].

Definition 5.2.18. For a compact, connected and orientable surface S we define the complexity of S as $\chi_-(S) := \max\{-\chi(S), 0\}$. For a disconnected surface this is naturally defined as

$$\chi_-(S) := \sum_{i=1}^{n} \chi_-(S_i)$$

where the surfaces S_i are the connected components of S.

We want to carry this notion from surfaces to homology classes in our 3-manifolds. To do this we need the following lemma. As always, let M be a compact, connected and orientable 3-manifold, and let H_* denote singular homology with integer coefficients.

Lemma 5.2.19. *Every class in $H_2(M, \partial M)$ may be represented by a compact, orientable and properly embedded surface.*

Proof. By Lefschetz Duality we have $H_2(M, \partial M) \cong H^1(M)$ and we recall from 2.1.19 the identifications $H^1(M) = \langle M, S^1 \rangle$. Therefore every class $\sigma \in H_2(M, \partial M)$ determines a map $M \to S^1$ unique up to homotopy; this allows us to deform it to a smooth function. Then the preimage of a regular value will be a properly embedded compact surface in M. When given the right orientation this will represent the class σ. $\square$

We are then legitimate to give the following definition.

Definition 5.2.20. For a subsurface N of ∂M and a class $\sigma \in H_2(M, N)$, we define the Thurston Norm

$$x(\sigma) := \min\{\chi_-(S) \mid (S, \partial S) \subset (M, N), [S] = \sigma\}$$

Notice that the duality $H_2(M, \partial M) \cong H^1(M)$ exploited in the above proof allows us to define a norm on cohomology as well: let $\varphi \in H^1(M)$, then we set

$$x(\varphi) := \min\{\chi_-(S) \mid (S, \partial S) \subset (M, N), [S] \text{ is dual to } \varphi\}$$

and call it the Thurston Norm of φ.

Remark 5.2.21. Of course we have that $x(\sigma) \geq 0$ for every $\sigma \in H_2(M, \partial M)$. Moreover Thurston proved that

1. $x(k\sigma) = |k|x(\sigma)$ for any $k \in \mathbb{Z}$
2. $x(\sigma + \tau) \leq x(\sigma) + x(\tau)$

This means that x defines a seminorm on $H_2(M, \partial M)$. By 1 above, we can extend x to $H_2(M, \partial M, \mathbb{Q})$ and then to $H_2(M, \partial M, \mathbb{R})$ by requiring it to be continuous. Another fundamental thing Thurston proved is that

3. $x(\sigma) = 0$ if and only if σ can be represented by a surface of non-negative Euler characteristic.

As a consequence, if M is irreducible and atoroidal (e.g. one of the generic pieces coming from a JSJ Decomposition), then every sphere and every torus can represent only the trivial class in $H_2(M, \partial M)$; this fact turns x into a norm on $H_2(M, \partial M, \mathbb{R}) \cong H^1(M, \mathbb{R})$, which motivates its name. This is of course the case we are interested in and to which we restrict from now on.

Thurston proved the following remarkable result about this norm in [Thu86].

Theorem 5.2.22 (Thurston). *If M is irreducible and atoroidal, then the unit ball $\mathcal{B}$ in the Thurston norm is a finite polyhedron.*

This is interesting because this allows for a very nice characterization of (co)homology classes associated to fibrations.

Remark 5.2.23. Let $p : M \to S^1$ be a fibration, i.e. the projection of a surface bundle structure on M; then it induces $p_* : \pi_1(M) \to \mathbb{Z}$; by the identification

$$H^1(X) \longleftrightarrow \langle X, S^1 \rangle \longleftrightarrow \mathrm{Hom}(\pi_1(X), \mathbb{Z})$$

of 2.1.19, we get a class in $H_2(M, \partial M) \cong H^1(M)$.

Definition 5.2.24. A class in $H_2(M, \partial M, \mathbb{Q}) \cong H^1(M, \mathbb{Q})$ is called a fibered class if it has some multiple induced by a fibration $p : M \to S^1$.

Notice that by definition if a class is fibered than any rational multiple is again fibered. Therefore one can speak of a fibered ray. To state the promised characterization we need some additional definitions.

Definition 5.2.25. Let $F \subset \mathcal{B}$ be a face of the Thurston unit ball in $H^1(M, \mathbb{Q})$. The Thurston cone over F is the set $\{\lambda f | f \in F, \lambda > 0\}$. In the general case in which M is not irreducible atoroidal we define a Thurston cone to be either one of these cones or a maximal connected subset of $H^1(M, \mathbb{Q})$ on which x vanishes.

We then have this result of Thurston (see also Theorem 5.15 in [Cal07]).

Theorem 5.2.26 (Thurston)**.** *There exists a (possibly empty) collection of open top-dimensional faces of the Thurston unit ball in $H^1(M, \mathbb{Q})$ such that the fibered classes in $H^1(M, \mathbb{Q})$ are exactly the points contained in the Thurston cones over the faces in this collection.*

If the collection in the above statement is not empty, than the faces in it are called fibered faces and the cones over them are called fibered cones. This is justified by the fact that if a class sits on a top dimensional face of the Thurston ball in $H^1(M, \mathbb{Q})$ and is fibered, then every other class on the same face is also fibered and any of its multiples is fibered too.

5.2.4 Virtual Fibration of RFRS Manifolds

In [Ago08] Agol proves the following result.

Theorem 5.2.27 (Agol)**.** *Let M be a compact irreducible orientable 3-manifold with $\chi(M) = 0$. If $\pi_1(M)$ is RFRS then M virtually fibers over the circle.*

The idea of Agol's proof is this: suppose $S \subset M$ is an oriented surface which is non-separating, i.e. $M \setminus \setminus S$ is connected. Then S is the fiber of a fibration over S^1 if and only if $M \setminus \setminus S \cong S \times I$. Otherwise $M \setminus \setminus S$ admits a non trivial JSJ Decomposition in which we find some pieces which are the product of a surface and an interval and pieces which are not products of this kind. Agol's RFRS condition allows to pass to suitable covers of M in which these non-product pieces are somehow "killed".

We can find in the literature (at least) two approaches to this idea. The first is Agol's original paper ([Ago08]), where he uses the theory of least-weight taut normal surfaces, which has a strong combinatorial flavour; the second is due to a later paper of Friedl and Kitayama ([FrK12]) and exploits the properties of sutured manifolds. In this section we give a precise meaning to the above idea and describe the main steps of the construction, focusing on the role of the RFRS condition.

Definition 5.2.28. A sutured manifold is a 3-manifold M with a decomposition of its boundary into oriented submanifolds $\partial M = R_- \cup \gamma \cup R_+$, where

- γ is a disjoint union of annuli (the sutures),
- R_- and R_+ are disjoint subsurfaces of ∂M such that for each component A of γ we have that $A \cap R_-$ is a boundary of both A and R_-, and the same for R_+,
- R_- and R_+ are oriented so that they induce the same orientation on each component of γ.

When the above conditions are satisfied we also say that (M, R_-, R_+, γ) is a sutured manifold.

The most basic (but nevertheless central) examples are the following.

Example 5.2.29. Let S be a surface and let $M = S \times I$. Then

$$(M, S \times \{-1\}, S \times \{1\}, \partial S \times I)$$

is a sutured manifold when things are endowed with the obvious orientations. We call such a manifold a product sutured manifold.

Example 5.2.30. Let M be a closed 3-manifold and $S \subset M$ a properly embedded surface. Then we get an associated sutured manifold

$$M(S) := (M \setminus (S \times I), S \times \{-1\}, S \times \{1\}, \varnothing)$$

We explicitly observe that S is the fiber of a fibration over the circle if and only if $M(S)$ is product sutured manifold.

The last example can be generalized to the case of non empty boundary by taking care of the boundary data. The idea is that if we choose the surface nice enough (with respect to the boundary) then $M \setminus \setminus S$ will be endowed with a canonical structure of sutured manifold and we say that it is a sutured manifold obtained by sutured decomposition along S; then the theory goes on mimicking the theory of Haken manifolds and constructing a so-called sutured hierarchy. More details can be found in [Sch90] and [FrK12]. We are especially interested in Theorem 3.2 of [FrK12], which can roughly be stated as follows.

Theorem 5.2.31. *Let (M, R_-, R_+, γ) be an irreducible sutured manifold. Then there exists a product sutured submanifold $P \subseteq M$, unique up to isotopy, such that any other product submanifold can be isotoped into P.*

In other words this means that a sutured manifold structure on a 3-manifold gives rise to a "maximal product core", essentially unique. This justifies the intuition behind the following definition (and also proves that it is well posed).

Definition 5.2.32. Let (M, R_-, R_+, γ) be an irreducible sutured manifold and let P the product sutured submanifold of the previous theorem. We call a window of M any component of P and a gut any component of $M \setminus \setminus P$.

The guts of M are the pieces we are left with after removing the product part of it. They carry non trivial topological information and are an obstruction to M being a surface bundle over the circle. The point of Agol's work is that there are suitable covering manifolds where these pieces can be simplified; these coverings are indeed chosen among those induced by a

RFRS series for the fundamental group. The Thurston norm is used in this procedure in order to choose the appropriate surfaces to cut along and to detect fibered classes (in the light of Theorem 5.2.26). This is Agol's original statement.

Theorem 5.2.33 (Agol)**.** *Let M be connected orientable irreducible 3-manifold with $\chi(M) = 0$ and $\pi_1(M)$ RFRS. If $f \in H^1(M)$ is a non trivial and non fibered class then there is a finite cover $p : N \to M$ such that $p^*f \in H^1(N)$ sits in the cone over the boundary of a fibered face of the Thurston ball $\mathcal{B}$ of N.*

The proof is quite technical and we refer the reader to the original paper of Agol [Ago08], or [FrK12] for a more detailed exposition. We just say a few words to give some ideas of why the purely algebraic RFRS condition should have something to do with this geometric procedure of cutting and covering. The RFRS condition consists of two parts:

- residual finiteness, which is related to the problem of embedding immersed compact subsets into suitable covering spaces,
- rational solvability, which has something to say about "homological largeness" of covering spaces.

We begin with the first one.

Definition 5.2.34. A group G is residually finite if $\forall\, g \in G, g \neq 1$ there is $H \leq G$ of finite index such that $g \notin H$.

The interest of this definition lies in the following result (see Lemma 1.3 in Scott's paper [Sco78]).

Lemma 5.2.35. *Let X be a Hausdorff topological space with regular covering space $\widetilde{X}$ and covering group G. Then G is residually finite if and only if for every compact subset $K \subset \widetilde{X}$ we can find an intermediate finite covering space $\widetilde{X} \to X' \to X$ such that C projects homeomorphically onto X'.*

The property of being residually finite is equivalent to saying that the intersection of all the finite index subgroups of G is trivial; therefore we see that a RFRS group is in particular residually finite. For the applications in 3-manifold topology we are especially interested in the case K is either a compact subsurface of our manifold, for example one of the immersed surfaces obtained by Kahn and Markovic, or some of the guts components induced by a sutured manifold structure.

Finally we turn to the rational solvability part and the usefulness of working with rational coefficients.

Definition 5.2.36. Let $p : \widetilde{X} \to X$ a covering projection of degree n. We define a map at the level of singular chains over any PID R as follows. Let $\sigma : \Delta^k \to X$ be a singular k-simplex and let $\{\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n\}$ be its lifting to $\widetilde{X}$. Define the so-called transfer homomorphism as

$$t : C_k(X, R) \to C_k(\widetilde{X}, R), \sigma \to \sum_{i=1}^{n} \widetilde{\sigma}_i$$

It is clear that composing the transfer map with the covering projection gives a map from $C_k(X, R)$ onto itself which is just multiplication by $n = deg(p)$. Moreover the transfer map commutes with the boundary operator, therefore we get a pair of maps in homology (which we still call p and t)

$$H_k(X, R) \xrightarrow{t} H_k(\widetilde{X}, R) \xrightarrow{p} H_k(X, R)$$

whose composition $pt : H_k(X, R) \to H_k(X, R)$ is just multiplication by n on $H_k(X, R)$. As a result we have that $\ker(pt)$ consists of torsion elements in $H_k(X, R)$ whose order divides n.

Remark 5.2.37. Working with coefficients in $R = \mathbb{Q}$ is just the same as working with integral homology modulo torsion; this implies that if we have a finite covering $p : \widetilde{X} \to X$ then the above map $pt : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$ is injective, just because $\mathbb{Q}$-homology groups are torsion free. But then also $t : H_k(X, \mathbb{Q}) \to H_k(\widetilde{X}, \mathbb{Q})$ in necessarily injective. This can informally be stated saying that the rational homology of $\widetilde{X}$ is larger than that of X. By duality the same holds for cohomology.

We apply this machinery in the case X is a closed hyperbolic 3-manifold M which has a decomposition into windows and guts as described above. Quoting Agol's original paper:

> The idea is to produce a complexity of the guts, and use the RFRS condition to produce a cover of M to which a component of the guts lifts and for which we can decrease the complexity of the guts, by "killing" it using non-separating surfaces coming from new homology in this cover.

The complexity Agol refers to is (a slightly modified version of) an invariant introduced by Gabai in [Gab83] which is defined for sutured manifolds and takes its values in a linearly ordered set in which strictly descending chains are finite. This means that if we are able to iteratively find finite covers where this complexity drops then in the end we are left with no guts, that is we have found a finite cover which is a surface bundle.

Bibliography

[Ago08] I. Agol. *Criteria for Virtual Fibering*, preprint arXiv: math/0707.4522 [math.GT], 2008.

[Ago12] I. Agol. *The Virtual Haken Conjecture*, preprint arXiv: math/1204.2810 [math.GT], 2012.

[AFW12] M. Aschenbrenner, S. Friedl, H. Wilton. 3-*manifold Groups*, preprint arXiv: math/1205.0202 [math.GT], 2012.

[BP92] R. Benedetti, C. Petronio. *Lectures on Hyperbolic Geometry*, Springer, Berlin, 1992.

[BW10] N. Bergeron, D. Wise. *A Boundary Criterion for Cubulation*, preprint arXiv: math/0908.3609v3 [math.GT], 2010.

[Bon02] F. Bonahon. *Geometric Structure on* 3-*Manifolds*, Handbook of Geometric Topology (R. Daverman, R. Sher eds.), Elsevier, 93-164, 2002.

[BrH99] M.R. Bridson. *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.

[Cal07] D. Calegari. *Foliations and the Geometry of* 3-*Manifolds*, Oxford University Press, Oxford, 2007.

[CaC00] A. Candel, L. Conlon. *Foliations*, American Mathematical Society, Providence, 2000.

[Cha06] R. Charney. *An Introduction to Right-angled Arting Groups*, preprint arXiv: math/0610668v1 [math.GR], 2006.

[Dav08] M.W. Davis. *The Geometry and Topology of Coxeter Groups*, Princeton University Press, Princeton, 2008.

[DJ00] M.W. Davis, T. Januszkiewicz. *Right-angled Artin Groups are commensurable with Right-angled Coxeter Groups*, Journal of Pure and Applied Algebra, 153, 229-235, 2000.

[FrK12] S. Friedl, T. Kitayama. *The Virtul Fibering Theorem for* 3-*manifolds*, preprint arXiv: math/1210.4799 [math.GT], 2012.

[Gab83] D. Gabai. *Foliations and the Topology of* 3-*manifolds*, Journal of Differential Geometry 18, no. 3, 445-503, 1983.

[Gro87] M. Gromov. *Hyperbolic Groups*, Essays in Group Theory (S.M. Gersten, ed), Springer, MSRI Publications 8, 75-263, 1987.

[HaW08] F. Haglund, D. Wise. *Special Cube Complexes*, Geometric and Functional Analysis 17, no. 5, 1551-1620, 2008.

[Hat02] A. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.

[Hat] A. Hatcher. *Notes on Basic 3-Manifold Topology.*

[Hem76] J. Hempel. 3-*Manifolds*, Annals of Mathematical Studies, vol. 86, Princeton University Press, Princeton, 1976.

[HW99] T. Hsu, D. Wise. *On Linear and Residual Properties of Graph Products*, Michigan Mathematical Journal, 46 (2), 251-259, 1999.

[Jac77] W. Jaco. *Lectures on Three-Manifold Topology*, CBMS Regional Conference Series in Mathematics, no. 43, Providence, 1977.

[Hae91] A. Haefliger. *Complexes of Groups and Orbihedra*, Group Theory from a Geometrical Viewpoint (Trieste, 1990), World Scientific Publishing, River Edge, NJ, 504-540, 1991.

[KM09] J. Kahn, V. Markovic. *Immersing Almost Geodesic Surfaces in a Closed Hyperbolic Three Manifold*, preprint arXiv: math/0910.5501v5 [math.GT], 2009.

[KS77] R.C. Kirby, L.C. Siebenmann. *Foundational Essays on Topological Manifolds, Smoothings and Triangulations*, Princeton University Press, 1977.

[Kob13] T. Koberda. *Right-angled Artin Groups and their Subgroups*, notes for a course at Yale University, 2013.

[LyS77] R. Lyndon, P. Schupp. *Combinatorial Group Theory*, Springer, Berlin, 1977.

[Sag95] M. Sageev. *Ends of Group Pairs and Non-positively Curved Cube Complexes*, Proceedings of the London Mathematical Society (3), 71 (3), 585-617, 1995.

[Sag97] M. Sageev. *Codimension-*1 *Subgroups and Splitting of Groups*, Journal of Algebra, 189 (2), 377-389, 1997.

[Sch90] M.G. Scharlemann. *Lectures on the Theory of Sutured* 3*-manifolds*, Korea Advanced Institute of Technology Workshop Proceedings, 25-46, 1990.

[Sco78] P. Scott, *Subgroups of Surface Groups are Almost Geometric*, Journal of the London Mathematical Society, Second Series 17, no. 3, 357-381, 1978.

[Sco83] P. Scott, *The Geometries of* 3*-Manifolds*, Bullettin of the London Mathematical Society, 15, 401-487, 1983.

[Thu86] W. P. Thurston. *A Norm for the Homology* 3*-manifolds*. Memories of the American Mathematical Society, 339, 99-130, 1986.

[Thu] W. P. Thurston. *The Geometry and Topology of Three-manifolds.*

[Thu82] W. P. Thurston. *Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry*, Bullettin of the American Mathematical Society, vol. 6, no. 3, 1982.

[Wil02] H. Wilton. *The* 3*-dimensional Geometrisation Conjecture of W.P. Thurston*, online notes, 2002.

[Wi12] D. Wise. *From Riches to Raags: 3-Manifolds, Right-Angled Artin Groups, and Cubical Geometry*, CBMS Regional Conference Series in Mathematics, no. 117, Providence, 2012.

[Wi11] D. Wise. *The Structure of Groups with a Quasiconvex Hierarchy*, preprint, 2011.

Acknowledgements

I want to thank my advisor, Stefano Francaviglia, for the enthusiasm, the patience, the trust and the encouragement he has shown me during the preparation of this thesis. I also want to thank Rita Fioresi and Luca Migliorini for all the time and energy they have spent in teaching me geometry and algebra, both inside and outside the teaching hours. A special thanks also goes to Alberto Parmeggiani for his guidance and his precious advices throughout my studies.

I also want to thank the institutions that have given me some kind of financial support during these years, as AILA, the Departments of Mathematics of Bologna and Pisa and the Collegio Superiore of Bologna.